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Markoff Cascades with General Source Terms.

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Summary. — The longitudinal development of the multiplicative cascade, considered as a generalized Markoff process, is presented for the case in which the source is not necessarily a δ-function of space or energy. Last collision diffusion equations are established and formally solved, and practical applications are discussed. In addition to its « traditional » application to cosmic radiation theory the Markoff cascade has important applications to the theory of nuclear reactors.

1. – Introduction.

Considerable advances have been made in recent years in applying the theory of Markoff processes to predicting the behaviour of multiplying systems. Examples of such systems are legion in both the physical and biological sciences. In physics perhaps the best-known example is that of a « cascade » of particles originating from a single high energy particle in passing through matter. To a lesser extent the theory of the cascade resulting from a point source of many particles having a spectrum of energy has also been developed. Much of this work has been addressed to the study of cosmic ray showers originating from high energy particles incident on the top of the earth's atmosphere. The longitudinal and lateral development of such showers has been analysed in numerous papers, and some elegant mathematical formalisms proposed for calculating the probability distributions for various kinds of particles of various

energies and at different points in the shower. A fairly complete bibliography of this work has been given by GARDNER *et al.* [1].

While the problem of the cascade originating from a single particle source may now be said to be solved in principle there remains much to be done in devising numerical methods and automatic computer programmes which will enable one to extract from the general integro-differential diffusion equations and their invariably intricate solutions, specific quantitative results suitable for comparison with experimental data in particular cases. Some such numerical methods have already been proposed, see, for example, ref. [1]; also GARDNER [2-4].

By comparison with the plethora of publications on cascade theory with a point source, the theory of a cascade developing from a spatially distributed source seems to have received scant attention to date. To the author's knowledge only three publications have appeared in this field [5-7]. These deal with the so-called ionization cascade, *i.e.* the cascade of knock-on electrons originating from the track of a fast ionizing particle passing through an absorber. Such a track will constitute a line source of primary knock-ons each of which may be a point source for a multiplicative process analogous to the nucleon cascade [1]. A fast primary knock-on will ionise an atom producing thereby secondary electrons which may in turn eject further electrons, the process terminating when all its end products are either too slow to ionise further or else have escaped from the medium or been absorbed by it.

It is not difficult to envisage other kinds of line source cascades: the « jets » originating from the traversal of a nucleus by an ultra-high-energy nucleon are one example [8]; another example, many decades lower in energy would be the « line source » of neutrons (usually in the form of an antimony-beryllium tube) installed in a nuclear reactor to maintain the chain reaction at zero power during shut down.

It is the purpose of the present paper to show how an established method of single source cascade theory—what may be briefly termed the « last collision equation » method—may be extended to yield a generalized formalism for describing a cascade originating from a source having a finite extent in space and/or energy. In a subsequent paper an alternative approach by the method of characteristic functionals is discussed.

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(1) J. W. GARDNER, H. GELLMAN and H. MESSEL: *Nuovo Cimento*, **2**, 58 (1955).
(2) J. W. GARDNER: *Nuovo Cimento*, **5**, 1368 (1957).
(3) J. W. GARDNER: *Nuovo Cimento*, **7**, 10 (1958).
(4) J. W. GARDNER: *Proc. Phys. Soc.* **5**, 205 (1960).
(5) J. W. GARDNER: *Phys. Rev.*, **94**, 764 (1954).
(6) J. W. GARDNER: *Proc. Phys. Soc.*, A **68**, 1072 (1955).
(7) J. E. MOYAL: *Nucl. Phys.*, **1**, 180 (1956).
(8) F. C. ROESLER and C. B. A. MCCUSKER: *Nuovo Cimento*, **10**, 127 (1953).

2. – Basic formulation.

In this section we shall set up the general equations describing the development of an extended source cascade containing only one kind of multiplying particle (Furry cascade). This last restriction is not an essential one but is made because the generalization to many types of particles would require much tedious reproduction of almost identical formulae without introducing any new principle. By the same token, we shall limit ourselves to a one-dimensional treatment and to non-ionizing particles (*e.g.* neutrons), since the introduction of refinements at this stage would only tend to obscure the basic development. Once the basic longitudinal development of the extended source cascade has been mathematically formulated, such things as the lateral spread and the introduction of ionizing particles can be accommodated in a manner completely analogous to the method used for their incorporation into the single-source cascade.

Let us postulate a source function $S(E, z)$ representing the probability of a primary particle's being «emitted» at the point $\xi = E$, $\zeta = z$ in the space. generated by the stochastic variables ξ, ζ. These variables we shall take for the sake of definiteness to be energy and depth, although in some applications they might more conveniently stand for, say, momentum and time. For a point source consisting of a single primary of energy E_0 at zero depth obviously $S(E, z)$ has the form $\delta(E - E_0)\,\delta(z)$.

We introduce the following definitions:

Let $\omega(\eta_i, \eta_j)\,\mathrm{d}\eta_i\,\mathrm{d}\eta_j\,\mathrm{d}x$ denote the probability for a primary particle of unit energy to have a collision (*) in distance $\mathrm{d}x$, resulting in two secondaries with energies in the intervals $\mathrm{d}\eta_i$ at η_i and $\mathrm{d}\eta_j$ at η_j. (We neglect ternary and higher order collisions which could however be accommodated by an obvious extension of this notation).

Let $p_n(\eta_1 \dots \eta_n; x)\,\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n$ be the differential probability that the postulated source distribution give rise at the depth x to n particles with energies in the ranges η_k, $\eta_k + \mathrm{d}\eta_k$ with $k = 1, \dots, n$ in any order.

Let $q_n(\eta_1 \dots \eta_n; x)\,\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n$ denote the differential probability that the postulated source distribution give rise at the depth x to n particles with energies specified as above, plus any number of particles with arbitrary energies, including if we wish the same energies specified above.

RAMAKRISHNAN [9] has discussed the concepts of the quantities p_n and q_n, and obtains (in effect) the following general mathematical result which is not

(*) We use the word «collision» only to fix our ideas: the multiplication could occur for example by spontaneous fission, or some other process not involving a collision in the conventional sense.

(9) A. RAMAKRISHNAN: *Proc. Camb. Phil. Soc.*, **46**, 595 (1950); **48**, 451 (1952).

restricted to any particular physical cascade or source distribution:

$$p_n(\eta_1, \dots \eta_n; x) = \frac{1}{n!} P(n, x) q_n(\eta_1, \dots \eta_n; x) , \tag{2.1}$$

with

$$P(n, x) = \int_0^\infty \dots \int_0^\infty p_n(\eta_1, \dots \eta_n; x)\, \mathrm{d}\eta_n \dots \mathrm{d}\eta_1 \tag{2.2}$$

Thus if either p_n or q_n is known the other may in principle be determined. However, experience with single source cascade theory has shown it to be more expedient in practice to derive diffusion equations for each of q_n and p_n and solve them directly; this principle applies *a fortiori* to the extended-source theory.

The usefulness of the quantity q_n, called by some authors the « product density », lies in its very simple relation with the practically useful function $\varphi_n(\eta, x)$ expressing the probability that the postulated source give rise at depth x to n particles with energies greater than η and any num berof particles with energies less than η. See, for example, reference (3). This relation is

$$T_n(\eta; x) = \int_\eta^1 \dots \int_\eta^1 q_n(\eta_1, \dots \eta_n; x)\, \mathrm{d}\eta_n \dots \mathrm{d}\eta_1 , \tag{2.3}$$

where $T_n(\eta, x)$ is the n-th factorial moment defined by:

$$T_n(\eta; x) = \sum_{a=0}^{\infty} \frac{(n+a)!}{a!} \varphi_{n+a}(\eta; x) . \tag{2.4}$$

Since the enormous complexity of the analytical expression for φ_n makes its direct calculation a formidable task, even with an electronic computer (3), the value of the q_n for obtaining the moments is immediately apparent; for one can get certain information about the shape of the distribution from its first few moments and may in fact be able to construct a reasonable approximation to φ_n from an expansion in the moments (2,3).

We now proceed to set up the last-collision diffusion equations for the p_n and q_n. Considering p_n first, we see that there are only two types of last collision which could give rise to n particles in the energy intervals η_i, $\eta + \mathrm{d}\eta_i$, $i = 1, \dots, n$ at depth x:

a) At some depth $x - \Delta x$, after the penultimate collision the cascade consists of $n-1$ particles of which $n-2$ have attained their final energies; the $(n-1)$-th particle then undergoes a collision attaining thereby its final energy and giving rise to the n-th particle, also with its final energy.

b) At some depth $x - \Delta x$, the cascade consists of $n-1$ particles *all* of which have attained their final energies; the source then emits the n-th

particle with appropriate energy. (Note we are excluding simultaneous emission of more than one particle by the source in the interval Δx; this places no practical restriction on the types of source which can be included in our formalism).

By the usual procedure of writing down the probabilities for a) and b), and allowing the interval Δx to become infinitesimal one has immediately the following diffusion equation for p_n:

$$(2.5)\qquad (\partial/\partial x + n\Omega)p_n = \sum_{C_n} p_{n-1}(\eta_n', \dots, \eta_{n-2}', \eta_{n-1}' + \eta_n'; x)\,\omega(\eta_{n-1}', \eta_n') + \\ + \sum_{C_n} p_{n-1}'(\eta_1', \dots, \eta_{n-1}'; x)\, S(\eta_n', x)\,,$$

where the C_n signify that the summations are taken over all possible choices of η_n' from the η_k, $k = 1, \dots, n$; and where:

$$(2.6)\qquad \Omega = \int_0^1 \omega(\eta_i, \eta_j)\,\mathrm{d}\eta_j + \int_0^1 \omega(\eta_i, \eta_j)\,\mathrm{d}\eta_i\,.$$

Since there is no loss of generality in assuming $\omega(\eta_i, \eta_j)$ to be symmetrical in η_i and η_j, the definition of Ω becomes

$$(2.6a)\qquad \Omega = 2\int_0^1 \omega(\eta_i, \eta_j)\,\mathrm{d}\eta_j = 2\int_0^1 \omega(\eta_i, \eta_j)\,\mathrm{d}\eta_i\,.$$

Rather than proceed immediately to the formal solution of (2.5) it is convenient first to establish the diffusion equation for q_n, and then consider solutions for p_n and q_n together. In the case of the quantity q_n it is easily seen that, in addition to last collisions of the type a) and b) above, we have a further possibility c) as follows:

c) At some depth $x - \Delta x$, after the penultimate collision the cascade consists of $n-1$ particles in the energy intervals η_i, $\eta_i + \mathrm{d}\eta_i$, $i = 1, \dots, n-1$, plus any number of other particles with arbitrary energies. One of these other particles then undergoes a collision giving rise to the n-th particle in the energy interval η_n, $\eta_n + \mathrm{d}\eta_n$.

Considering all types of last collision a), b) and c), and allowing Δx to become infinitesimal as before, we have the following diffusion equation for q_n:

$$(2.7)\qquad (\partial/\partial x + n\Omega)\,q_n = \sum_{C_n} q_{n-1}(\eta_1', \dots, \eta_{n-2}', \eta_{n-1}' + \eta_n'; x)\,\omega(\eta_{n-1}', \eta_n') + \\ + \sum_{C_n} \int_0^1 q_{n-1}(\eta_1', \dots \eta_{n-1}', u; x)\, 2\omega(u - \eta_n', \eta_n')\,\mathrm{d}u + \sum_{C_n} q_{n-1}(\eta_1', \dots \eta_{n-1}'; x)\, S(\eta_n', x)\,.$$

In the next section we discuss the solution of (2.5) and (2.7).

3. – Solution of basic equations.

In the course of our discussion we shall make repeated use of certain integral transforms of the functions p and q. Denoting either p or q by $f(\eta_1 \dots \eta_n; x)$ we define the Laplace transform of f with respect to x:

$$L_f(\eta_1 \dots \eta_n; \lambda) = \int_0^\infty \exp[-\lambda x]\, f(\eta_1, \dots, \eta_n; x)\,\mathrm{d}x\,, \tag{3.1}$$

the n-fold Mellin transform with respect to $\eta_1, \dots, \eta_n$:

$$M_f(s_1, \dots, s_n; x) = \int_0^1 \dots \int_0^1 \eta_1^{s_1} \dots \eta_n^{s_n} f(\eta_1, \dots, \eta_n; x)\,\mathrm{d}\eta_1 \dots \mathrm{d}\eta_n\,, \tag{3.2}$$

and in particular the Mellin-Laplace transform

$$N_f(s_1, \dots s_n; \lambda) = \int_0^1 \dots \int_0^1 \eta_1^{s_1} \dots \eta_n^{s_n} L_f(\eta_1, \dots \eta_n; \lambda)\,\mathrm{d}\eta_1 \dots, \mathrm{d}\eta_n\,. \tag{3.3}$$

In order to secure unambiguous inverse transformations we must expect L_f and M_f to be analytic within the strips D_λ and D_s along the imaginary axes of the λ and s planes respectively.

If we first subject (2.5) and (2.7) to the Laplace transformations we obtain a new set of inhomogeneous integral equations of which the following is typical:

$$\begin{aligned}(\lambda + \Omega) L_q(\eta_1, \dots, \eta_n; \lambda) = \sum_{C_n} L_q(\eta_1, \dots, \eta_{n-1}; \lambda)\,\cdot \\ \cdot \left\{F(\eta_n, \lambda) + \int K(\eta_1, \dots, \eta_{n-1}; u)\,\mathrm{d}u\right\}.\end{aligned} \tag{3.4}$$

Here the kernel function K is defined in such a way as to include the first two terms on the right side of (2.7), and the coefficient F arises from the source term S, considered here as a known quantity.

The possibility of a successful application of Mellin transformations on equations such as (3.4) depends entirely on the mathematical structure of the kernel functions $K(\eta_1, \dots, \eta_{n-1}, u)$. For example, it is well known that if $u\,K(\eta_1, \dots, \eta_{n-1}, u)$ can be represented by homogeneous functions of order zero the Mellin transformation reduces equation (3.4) to a set of linear algebraic equations and thus leads directly to the solution of the problem. Unfortunately

this condition is fulfilled in only a relatively limited class of physical problems, as has been pointed out for example by OLBERT and STORA [10]. These authors have calculated the average numbers, but not the fluctuations, for the general cascade originating from a point source with an arbitrary energy spectrum. In this case the kernel K is a function of only two energies (E', E corresponding to the producing and produced particles) and the above homogeneity condition therefore becomes:

$$K(E, E') = \left(\frac{1}{E'}\right)\Psi(E/E')\,, \tag{3.5}$$

where Ψ is some arbitrary function of E/E'.

Although, as has been said, condition (3.5) is physically rather restrictive it has been shown by OLBERT and STORA that for a wide class of physical cascade models, including for example those of FERMI [11] and of BELEN'KIJ and LANDAU [12], the following useful relation holds true:

$$\int_0^\infty E^s K(E, E')\,\mathrm{d}E = G(s)\,E'^{\sigma(s)}\,, \tag{3.6}$$

where $G(s)$ and $\sigma(s)$ are some suitable functions of the (complex) Mellin transform parameter s, independent of E'. One may regard (3.6) as a relation which selects among all possible functions of two variables, E and E', those whose Mellin transforms with respect to E are proportional to an arbitrary power of E'; clearly (3.6) embraces a rather general class of kernels.

The corresponding relations to (3.6) for a kernel of n parameters are readily seen to be:

$$\int_0^\infty \eta_k^{s_k} K(\eta_1, \ldots, \eta_{n-1}; u)\,\mathrm{d}\eta_k = G(s_k)\,u^{\sigma(s_k)}\,, \qquad k = 1, \ldots, n-1. \tag{3.7}$$

The $G(s_k)$ now involve the η's but are independent of u. Although the generalization from (3.6) to (3.7) presents no conceptual difficulty the computational labour of obtaining a practical solution has clearly been increased by a factor $\sim n$. Indeed it is not to be expected that closed analytic solutions for equations such as (3.4) are to be obtained in any but the simplest physical cases, and in most cases it will be feasible to obtain numerical results only by more or less extensive use of automatic computing facilities.

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[10] S. OLBERT and R. STORA: *Ann. Phys.*, **1**, 247 (1957).

[11] E. FERMI: *Phys. Rev.*, **81**, 683 (1951).

[12] S. Z. BELEN'KIJ and L. D. LANDAU: *Suppl. Nuovo Cimento*, **3**, 16 (1956).

Postponing the discussion of its practical evaluation to a later section we now proceed with the formal solution of (3.4) supplemented by (3.7). Applying Mellin transforms of the type (3.2) to both sides of (3.4) we obtain by virtue of (3.7) a new set of equations:

$$(3.8)\qquad (\lambda + \Omega) N_q(s_1, \dots, s_n; \lambda) = \\ = \sum_{C_n} N_q(s_1, \dots, s_{n-1}; \lambda)\{N_s(s_n, \lambda) + G(s_k) N_u(s_1, \dots, \sigma[s_k], \dots, s_{n-1}; \lambda)\},$$

with $k = 1, \dots, n-1$, and where N_s, N_u denote the appropriate Mellin-Laplace transforms on S and u.

A set of equations such as (3.8) is most conveniently solved by matrix methods. In matrix notation equations (3.8) may be written:

$$(3.9)\qquad \boldsymbol{N}(s_1, \dots, s_n; \lambda) - \sum_{C_n} \boldsymbol{A}(s_k, \lambda) \boldsymbol{N}(s_1, \dots, s_{n-1}; \lambda) = 0\,,$$

where the $\boldsymbol{N}$ matrices are constructed from the N_q according to familiar rules (see *e.g.* references [3] and [10]) and where:

$$(3.10)\qquad \boldsymbol{A}(s_k, \lambda) = \{\boldsymbol{N}_s(\sigma_k, \lambda) + G(s_k) \boldsymbol{N}_u(\sigma(s_k), \lambda)\}/(\lambda + \Omega)\,.$$

(The passive arguments of N_s and N_u are omitted for brevity).

The solution of (3.9) may now be accomplished, as in reference [10], by postulating an infinite sequence of functions σ_1, σ_2, ... such that $\sigma_1 = \sigma(s_k)$, $\sigma_2 = \sigma[\sigma(s_k)]$, etc.

Using this notation and an iteration on (3.9), supplemented by (3.10) one finds the following infinite series:

$$(3.11)\qquad \boldsymbol{N}(s_1, \dots, s_n; \lambda) = \boldsymbol{N}_s(s_k, \lambda) + \sum_{j=1}^{\infty} \boldsymbol{B}_j(\sigma_k, \lambda) \boldsymbol{N}_s(\sigma_j, \lambda)\,,$$

where

$$(3.12)\qquad \boldsymbol{B}_j(\sigma_k, \lambda) = \boldsymbol{A}(s_k, \lambda) \boldsymbol{A}(\sigma_1, \lambda) \dots \boldsymbol{A}(\sigma_{j-1}, \lambda)\,.$$

The series defined by the right side of (3.11) represents the required solution of (3.9) provided it converges uniformly for all values of s_k $(k = 1, 2, \dots, n-1)$ within a pertinent strip D_s along the imaginary axis of the complex s-plane. Whether or not the series converges depends of course on the behaviour of the function σ_j and $G(\sigma_j)$; this question has to be investigated separately in each individual case. Two such cases, corresponding to rather simple source terms are considered in the next section.

Assuming that the series converges the remaining task leading to $q_n(\eta_1, \dots, \eta_n; x)$ consists then in the inversion of the Mellin-Laplace transforms, involving the evaluation of complex integrals of the form:

$$\frac{1}{(2\pi i)^{n+1}} \int\limits_{u_n - i\infty}^{u_n + i\infty} \dots \dots \int\limits_{u_1 - i\infty}^{u_1 + i\infty} \quad \int\limits_{\lambda_1 - i\infty}^{\lambda_1 + i\infty} N_q \frac{\mathrm{d}\eta_n}{\eta_r^{s_n+1}} \dots \frac{\mathrm{d}\eta_1}{\eta_1^{s_1+1}} \exp\left[\lambda x\right] \mathrm{d}\lambda \,. \tag{3.13}$$

The exact evaluation of such integrals is possible only for some special, mathematically favourable cases, and in general one must resort to approximate methods such as that of steepest descent.

To conclude the present section we remark that although for the sake of definiteness, our discussion has been limited to the solution of the diffusion equations for q_n, the solution of those for p_n follows in an exactly similar manner. Thus p_n is given by an inverse Mellin-Laplace transform identical with (3.13), save that N_q is replaced by N_p, where the latter quantity is obtained from the solution of a matrix equation similar to (3.11). In fact the matrix equation for the N_p is formally identical with (3.11), although the specific form of the G functions will be different, and indeed somewhat simpler. This stems from the fact that the right side of the diffusion equation (2.5), for p_n contains one less term than the right side of the corresponding equation, (2.7), for q_n, and this leads to a simpler kernel K in the transformed equation (3.4) and hence, via (3.7), to a simpler G function.

4. – Discussion.

Closed solutions to (3.8) may be expected only for mathematically favourable forms of the N_s and G, corresponding respectively to rather special source functions and collision cross-sections; and even in such cases the numerical evaluation of the solutions is likely to require automatic computing facilities except for the lowest values of n. However, as has been pointed out, in reference (2) for example, it is often to just such low values of n that physical interest attaches. For example, the quantity $q_n(\eta_1, \dots, \eta_n; x)$ for $n = 1$ is very simply related by means of (2.3) to the average number of particles above energy η at depth x; and for $n = 2$ the spread about this average is related to q_2.

Another quantity having a direct physical interpretation is the $P(n, x)$ defined by (2.2): in view of the integrations over all energies $P(n, x)$ is clearly the probability of finding at the depth x exactly n particles, regardless of their energies and order. As defined, $P(n, x)$ automatically satisfies the nor-

malization condition:

$$\sum_{n=0}^{\infty} P(n, x) = 1 . \tag{4.1}$$

Now the diffusion equation for P is obviously obtained from that for p (*viz.* (2.5)) by integrating both sides with respect to all energies. Moreover, in view of this elimination of the energy variables, the transformed equation for P, corresponding to (3.8), is obtained by the operation of a single Laplace transform on the diffusion equation. By analogy with (2.5) and (3.4) it is thus possible to write down immediately the Laplace-transformed equation for P_n:

$$(\lambda + n) L_P(n, \lambda) = L_P(n-1, \lambda) + L_{PS}(n-1, \lambda) . \tag{4.2}$$

Here L_{PS} signifies the Laplace transform of the product of P with the source term, and the other quantities in (4.2) are to be interpreted according to the definition (3.1). Closed solutions to (4.2) are possible for appropriate source terms. The energy spectrum of the source is obviously immaterial since P takes no account of energies, and since the particle multiplication per collision (namely two) is by definition independent of the particle energies (*).

Thus it is necessary to specify only the spatial dependence of S, and since we are restricting ourselves to a one-dimensional treatment it is clear that point sources, line sources, or combinations of both are the only ones admissible. For the single point source $S(x) = \delta(0)$ the solution of (4.2) is fairly trivial and yields the familiar result:

$$P(n, x) = e^{-x}(1 - e^{-x})^{n-1} . \tag{4.3}$$

For a series of point sources located at $x = x_1, x_2, \ldots, x_m$ the solution of (4.2) is also straightforward, although giving rise to somewhat cumbersome expressions if m is large. In abbreviated notation, the general solution for m point sources distributed as above may be written:

$$P_m(n, x) = \exp\,[X_m - x] \sum_{i=1}^{m} - F_i(n, x)/J_i(m) , \tag{4.4}$$

with

$$X_m = \sum_{j=1}^{m} x_j , \tag{4.5}$$

(*) For this reason also P is independent of the cross-section ω as a function of energies.

$$F_i(n, x) = \{1 - \exp[-x + x_i]\}^{n-1}, \tag{4.6}$$

$$J_i(m) = \begin{cases} \prod\limits_{\substack{k=1 \\ \neq i}}^{m} (\exp[x_i]) - (\exp[x_k]) & m > 1, \\ 1 & m = 1. \end{cases} \tag{4.7}$$

The above result has been applied to calculated numerical values of $P(n, x)$ for the following for source arrangements (in which all distances are in cascade units):

(*a*) Point sources at $x = 0,\ 1$

(*b*) Point sources at $x = 0,\ \frac{1}{2},\ 1$

(*c*) Point sources at $x = 0,\ \frac{1}{3},\ \frac{2}{3},\ 1$

(*d*) Point sources at $x = 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1$.

In each case $P(n, x)$ was computed for $x = 2$ and 3 cascade units, and for n ranging up to 14. The results are shown in Figs. 1 and 2.

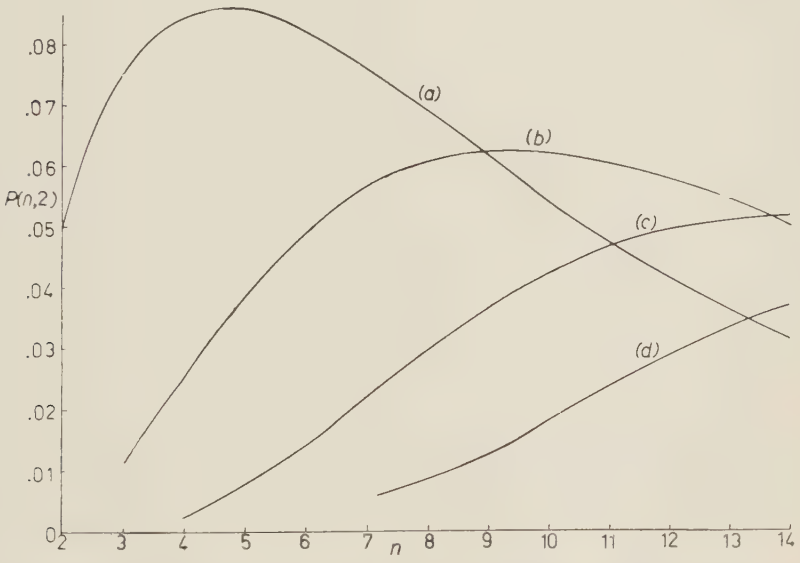

Fig. 1. – Probability $P(n, 2)$ of finding n particles at depth $x = 2$ due to point sources at: (*a*) $x=0, 1$; (*b*) $x=0, \frac{1}{2}, 1$; (*c*) $x=0, \frac{1}{3}, \frac{2}{3}, 1$; (*d*) $x=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Although one might expect to be able to simulate a line source by progressively increasing the number of point sources in a finite length it is clear from

Figs. 1 and 2 that m would have to be appreciably greater than 5 to give a satisfatory simulation. The case of a line source could in principle be treated by a numerical solution of (4.2) with the appropriate source term, *viz*:

$$S(x) = \begin{cases} \text{constant} & x_1 < x < x_2\,, \\ 0 & \text{all other } x\,. \end{cases} \tag{4.8}$$

However, as will be shown in a subsequent paper, the line source cascade is conveniently treated by an alternative method using characteristic functionals.

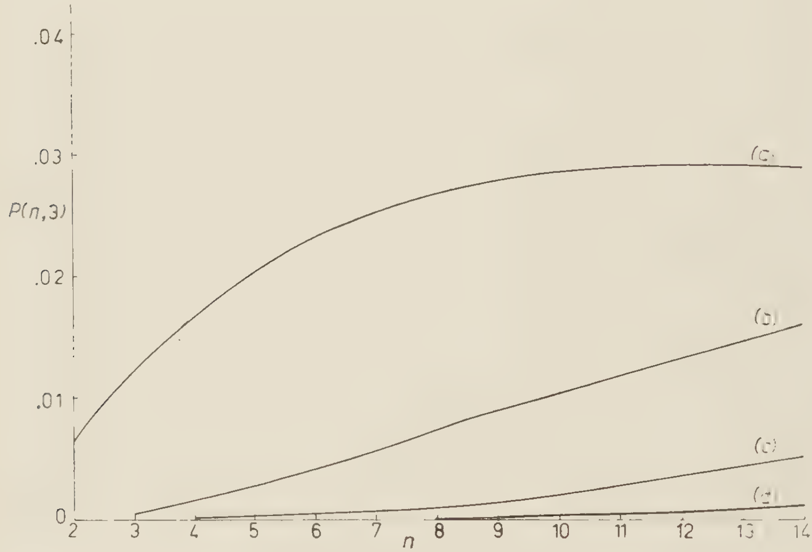

Fig. 2. – Probability $P(n, 3)$ of finding n particles at depth $x=3$ due to point sources at: (a) $x=0, 1$; (b) $x=0, \frac{1}{2}, 1$; (c) $x=0, \frac{1}{3}, \frac{2}{3}, 1$; (d) $x=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

As a second practical example we may instance the problem of finding the average number of particles above a given energy η at a depth x, arising from a single point source (at zero depth) of particles whose energies obey a power law spectrum. In such a case the source function would be defined by:

$$S(E, x) = \begin{cases} \delta(x)\, A E^{-(\gamma+1)}\,, & E > E_c\,, \\ 0\,, & E < E_c\,, \end{cases} \tag{4.9}$$

where A and γ are assumed to be (empirically) known quantities.

It is clear from the discussion at the beginning of this section that the required average number, $N(\eta, x)$ say, is given by:

$$\int_{\eta}^{1} q_1(\eta_1; x)\,\mathrm{d}\eta_1 = T_1(\eta, x) = N(\eta, x)\,, \tag{4.10}$$

so that our problem is to solve (2.7) for $n = 1$ and S given by (4.9), and then integrate the resulting expression for $q_1(\eta_1; x)$ with respect to η_1 between the limits $\eta \leqslant \eta_1 \leqslant 1$. Now in the present special case it may be readily verified that the Mellin-Laplace transform (3.3) applied to (2.7) yields, after a little re-arrangement and obvious changes in notation, the equation (47) of reference ([10]). This result was of course to be expected, since our general formalism in Section 2 contains no new assumptions about the basic Markoff processes responsible for cascade multiplication and must therefore yield the results of more limited « traditional » theories as special cases.

To obtain numerical results from the integration in (4.10) one must of course assume some model for the cross-section ω. Such models have been proposed by HEITLER and JÁNOSSY ([13]), by FERMI ([11]), and by BELEN'KIJ and LANDAU ([12]) and results based on the last two models are given in reference ([10]).

Even without considering further particular examples it is already evident that the generalization of source terms in the Markoff cascade has introduced an additional order of complexity in its mathematical formulation, not to mention solution. The solutions presented in Section 3 above are formal, amounting in effect to a set of instructions for mathematical operations on certain functions. These operations can be performed analytically only in the simplest cases, such as those discussed in the present section, and their accomplishment by numerical methods for less simple cases presents a fascinating challenge to modern automatic computing. The fact that the corresponding challenge has been largely met in the case of the point-source cascade encourages one to believe that computing development could in the next year or two (*) permit numerical solutions also in the case of general source terms.

Apart from its application to cosmic ray theory it is now realized that the Markoff cascade has important applications to nuclear reactor theory, for example in the calculation of neutron flux spectra and in shielding problems

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([13]) W. HEITLER and L. JÁNOSSY: *Proc. Phys. Soc.*, A **62**, 374 (1949).

(*) It would be out of place in a paper of this kind to speculate on the probable nature of these developments but it is of interest to note that a simple analogue computer is already being contemplated which may represent a significant advance over digital machines for the performance of certain types of Monte Carlo calculations. [See for example R. J. GOMPERTS: *Journ. Brit. Inst. Radio Engrs.*, **17**, 421 (1957).]

involving deep neutron penetration ([14]). This should provide a decisive stimulus to the necessary computing developments.

* * *

The author is most grateful to Miss DOROTHY AGAR for computational assistance in connexion with Figs. 1 and 2, and thanks the English Electric Co. Ltd. for having given permission to publish.

([14]) C. N. KLAHR: *Nucl. Sci. and Engrg.*, **3**, 269 (1958).

RIASSUNTO (*)

Si presenta lo sviluppo longitudinale della cascata moltiplicativa, considerata un processo di Markoff generalizzato, per il caso in cui la sorgente non sia necessariamente una funzione δ dello spazio e dell'energia. Si formulano le equazioni di diffusione dell'ultima collisione, che vengono formalmente risolte, e se ne discutono le applicazioni pratiche, oltre alla sua «tradizionale» applicazione alla teoria della radiazione cosmica la cascata di Markoff ha importanti applicazioni nella teoria dei reattori nucleari.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVI, N. 5 16 Giugno 1960

Measurements on Pick-Up Reactions in ^{31}P and ^{32}S.

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(ricevuto il 9 Febbraio 1960)

Summary. — Experimental results on angular distribution of deuterons emitted in reaction $^{31}P(n, d)^{30}Si$ and $^{32}S(n, d)^{31}P$ are presented. The reactions are due to 14.1 MeV neutrons and only the transition to the ground state is considered. The results are consistent with the simple interpretation that the reaction take place through a pure pick-up process with zero angular momentum transferred to the residual nucleus.

1. – Introduction.

The knowledge of the angular distribution of the particles emitted in a nuclear reaction at intermediate energy is extremely interesting since it almost always makes it possible to determine the mechanism by which the nuclear reaction itself occurs.

A case which is particularly clear is that of (n, d) reactions, for which a neutron is captured by the bombarded nucleus, and a deuteron is emitted. For these reactions, two possible interaction mechanisms can be considered:

1) direct interaction in which a peripherical proton is captured by the neutron to form the deuteron, without interaction with the core of the nucleus; this is the well-known pick-up reaction,

2) the formation of a compound nucleus and the subsequent evaporation of the deuteron.

It may also be asked if not both mechanisms are present. A conclusive reply to this problem can be given by the study of the angular distribution of the emitted deuterons.

In process (1) the angular distribution depends on the angular momentum transferred to the nucleus in the reaction, and this, once the angular momenta of the initial as well as of the final nucleus are known, can be calculated, showing well-pronounced maxima at well-defined angles; the detailed shape is characteristic of the value of the transferred angular momentum (1). For process (2), an isotropic or at least a smooth symmetric distribution with respect to 90° is foreseen (2).

2. – Description of measurements and results.

In this work we are presenting the measurement of the angular distribution of the deuterons emitted in the reactions $^{31}P(n, d)^{30}Si$ and $^{32}S(n, d)^{31}P$ which leave the residual nucleus in the ground state.

The existence of these reactions, with a large value of the cross-section, has recently been demonstrated (3). The total angular momentum of the ground states is known for all the nuclei involved in these reactions and it is therefore possible to calculate the angular distribution expected according to the pick-up theory; in both cases the transferred angular momentum is $l = 0$.

In the case of $^{31}P(n, d)^{30}Si$ the measurement was made at eight angles, between 7° and 135° with respect to the incidence direction of the neutrons, and at eleven angles, between 7° and 160°, in the case of $^{32}S(n, d)^{31}P$. At each angle, the whole energy spectrum was measured of the particles emitted by ^{31}P and ^{32}S following the bombardment of 14.1 MeV neutrons (Fig. 1 and 2).

The detector is the same as described in our previous works, consisting of a telescope of proportional and scintillation counters. The angular aperture is of about 10° at each angle.

It is easy to recognize in these spectra the peaks due to the deuterons of the maximum permitted energy, which leave the residual nucleus in the ground state. The total number of deuterons contained in every peak is therefore easily calculated, and these values are then corrected for the small differences

(1) S. T. Butler: *Nuclear Stripping Reactions* (New York, 1957).
(2) T. Ericson and V. Strutinski: *Nucl. Phys.*, **8**, 284 (1958).
(3) L. Colli, F. Cvelbar, S. Micheletti and M. Pignanelli: *Nuovo Cimento*, **13**, 868 (1959).

in the solid angles peculiar to every measurement. The results thus obtained are shown in Figs. 3 and 4, for ^{31}P and ^{32}S respectively.

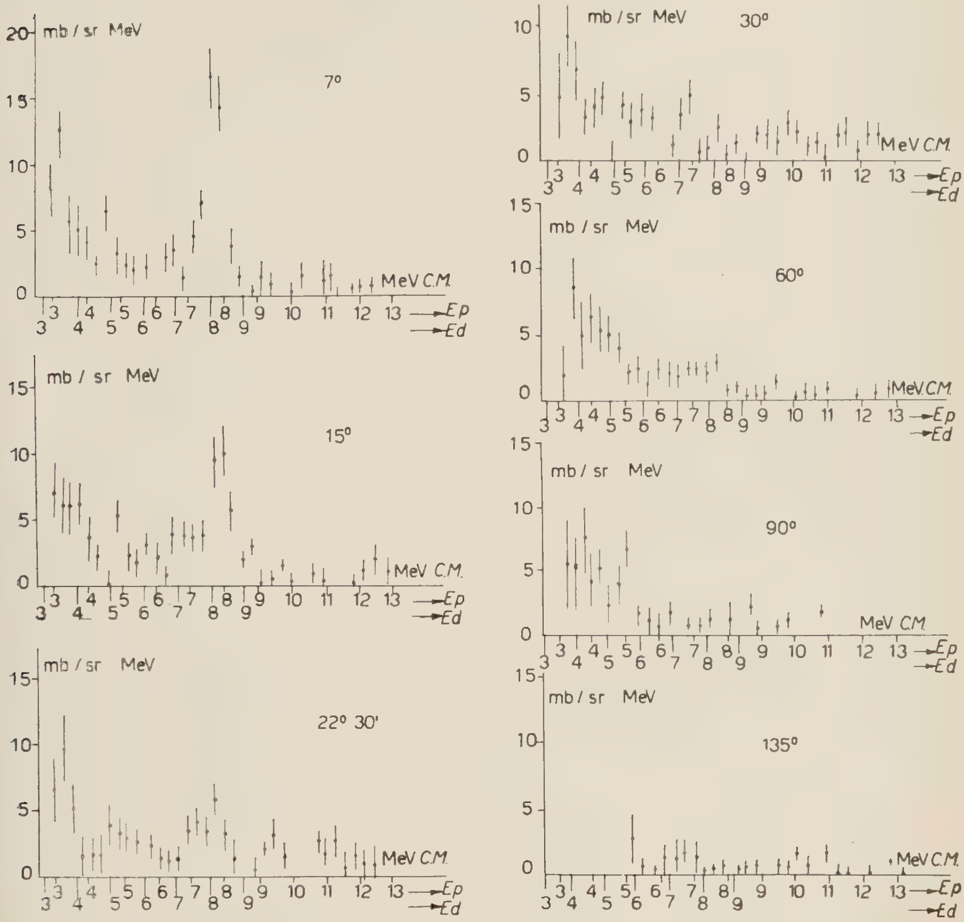

Fig. 1. – Spectra of (n, p) protons and (n, d) deuterons at various angles from ^{31}P: $E_n = 14.1$ MeV (lab.). The deuteron c.m. energy of the ground state peak is 8.05 MeV.

The two angular distributions, very similar to each other, present a strong anisotropy, with greater emission probability in the forward direction.

A contribution by evaporation is therefore certainly small; a small isotropic contribution can also be given by the protons of the (n, p) reactions, against which we did not discriminate.

Only a reaction of the pick-up type can account for an angular distribution similar to the one found. On the basis of the theory of these reactions, as developed by BUTLER, we have calculated the angular distribution for a pick

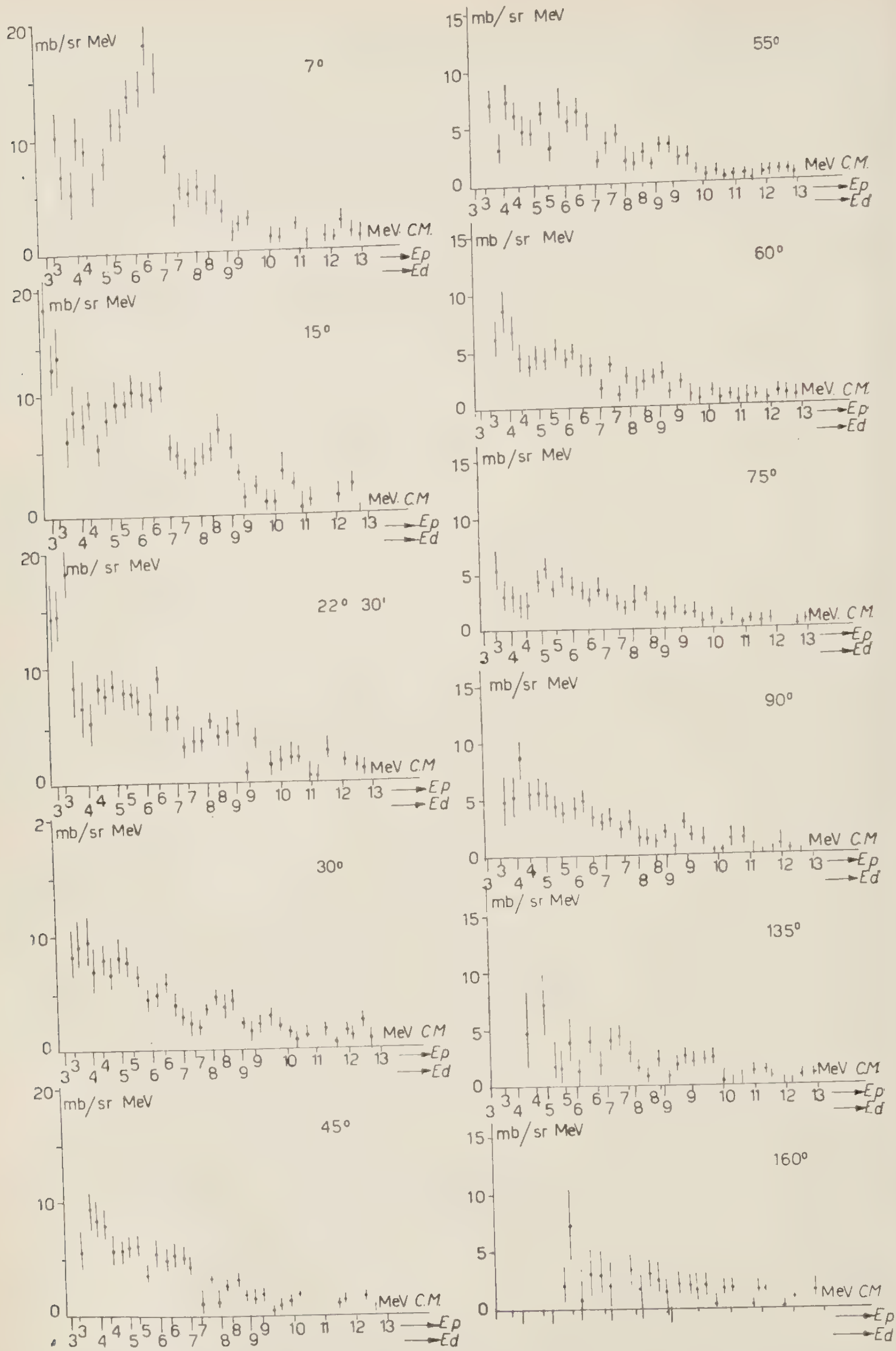

Fig. 2. – Spectra of (n, p) protons and (n, d) deuterons at various angles from ^{32}S; $E_n = 14.1$ MeV. The deuteron c.m. energy of the ground state peak is 6.65 MeV.

up reaction with an angular momentum transfer $l = 0$, as obtained in our two cases, and the agreement with the experiment was found very good when a value of $r = 5.53$ and $r = 5.57 \cdot 10^{-13}$ was assumed for the interaction radius

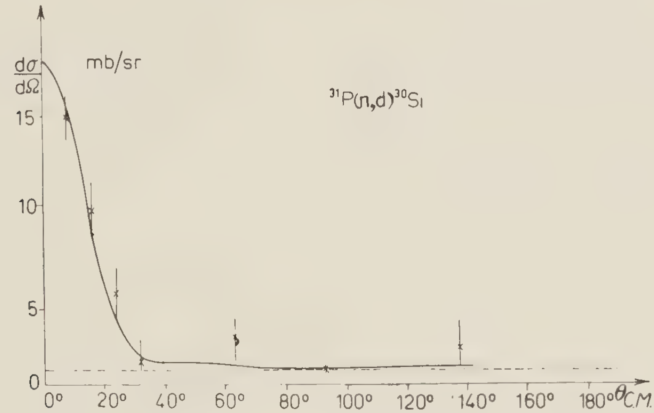

Fig. 3. – C.m. angular distribution of deuterons from the reaction ^{31}P(n, d)^{30}Si between ground states. Full curve: theoretical curve (Butler's theory) with $l=0$, $r=5.53 \cdot 10^{-13}$ cm, smeared for angular resolution.

and the correction due to Coulomb interaction of the captured proton is taken into account. The theoretical curves thus obtained, corrected for the finite angular resolution, are reported in the Figs. 3 and 4, with an arbitrary normalization.

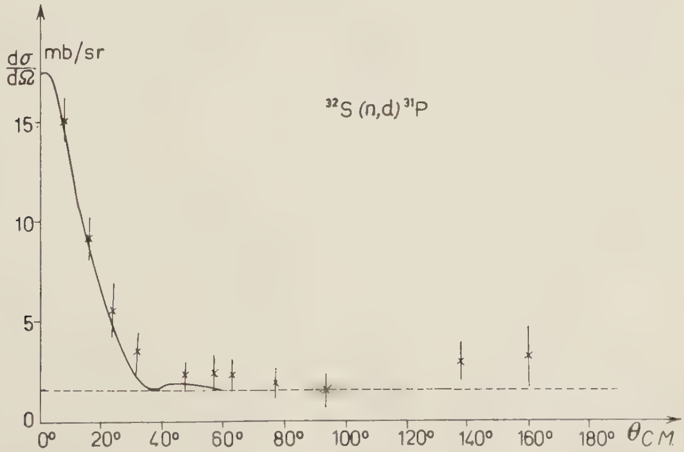

Fig. 4. – C.m. angular distribution of deuterons from the reaction ^{32}S(n, d)^{31}P between ground states. Full curve: theoretical curve (Butler's theory) with $l=0$, $r=5.57 \cdot 10^{-13}$ cm, smeared for angular resolution.

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The value of the radius chosen corresponds to the Gamow-Critchfield formula, which gives a good agreement with most of dip angular distributions

$$r = (1.22\,A^{\frac{1}{3}} + 1.7)\cdot 10^{-19}\ \text{cm}\,.$$

Values of the total cross-sections for these peaks after the subtraction of the small isotropic component are:

$$^{31}\text{P}(\text{n},\text{d})\,^{30}\text{Si} \qquad \sigma = (14.5 \pm 3)\ \text{mb}\,,$$

$$^{32}\text{S}\,(\text{n},\text{d})\,^{31}\text{P} \qquad \sigma = (14 \pm 4)\ \text{mb}\,.$$

The errors include: statistical error, geometrical imperfection, the error in target weight determination, and the error in the (n, p) scattering cross-section at 0° which has been used for normalization.

3. – Conclusion.

The experiments show that the (n, d) reactions here studied go through the pick-up mechanism and that they are in good agreement with Butler's calculations, notwithstanding the many approximations used in deriving the theory.

Following Butler's description, the l value given by the angular distribution corresponds to the l value of the orbit where the captured proton was found. The result $l = 0$ for both ^{31}P and ^{32}S is in agreement with the shell model assignment for these nuclei. Indeed ^{31}P has one s proton beside the closed $d_{\frac{5}{2}}$ shell, and ^{32}S has two s protons. Also interesting is the fact that these nuclei give the same cross-section value.

RIASSUNTO

In questo lavoro sono presentate le misure della distribuzione angolare dei deutoni emessi nelle reazioni ^{31}P(n, d)^{30}Si e ^{32}S(n, d)^{31}P. Queste reazioni sono state ottenute con neutroni da 14.1 MeV. Sono stati considerati soltanto i deutoni che corrispondono alla transizione allo stato fondamentale del nucleo residuo. I risultati sono in accordo con l'interpretazione di queste reazioni per mezzo di un processo di pick-up, del tipo descritto da Butler, corrispondente ad un trasferimento di momento angolare $l = 0$.

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Electron Pair Creation in $\pi^- + p$ Capture Reactions from Rest (*).

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(ricevuto il 23 Febbraio 1960)

Summary. — Electromagnetic corrections are calculated for the processes $\pi^- + p \to n + e^+ + e^-$ and $\pi^0 \to \gamma + e^+ + e^-$. These corrections depend on the c.m. energy of the pair. For very low c.m. energy the corrections are relatively large, and due mainly to the electron-positron Coulomb attraction; they fall rapidly with increasing c.m. energy to a minimum of 0.5%, then rise slowly to a value of 1.7% at the highest c.m. energy. For both processes, the correction to the total rate amounts to about 1%. In terms of these total rates, the Panofsky ratio is found to be

$$R_P = 0.594\, P[\pi^- + p \to n + \pi^0,\ \pi^0 \to \gamma + e^+ + e^-]/P[\pi^- + p \to n + e^+ + e^-]\,.$$

For the process $\pi^- + p \to n + e^+ + e^-$, the problem of empirically identifying the contributions arising from interaction via the longitudinal component of the electromagnetic field and from size effects is discussed. These contributions are expected to amount to only 2.0% and 0.8%, respectively, of the total rate. The process $\pi^- + p \to n + 2\gamma$ is also discussed briefly.

1. – Introduction.

The processes to be studied here are:

$$\pi^- + p \to n + e^+ + e^- \tag{1.1}$$

and

$$\pi^- + p \to n + \pi^0\,, \qquad \pi^0 \to \gamma + e^+ + e^-\,. \tag{1.2}$$

(*) A thesis submitted in partial fulfillment of the requirements for the Ph. D. degree in the Department of Physics at the University of Chicago.

(**) National Science Foundation Predoctoral Fellow, 1958-59. Now at Purdue University, Lafayette, Ind.

These are of interest because, first, the ratio of the two processes is directly related to the Panofsky ratio, $P[\pi^- + p \to n + \pi^0]/P[\pi^- + p \to n + \gamma]$, and, second, a detailed empirical analysis of process (1.1) would yield new information on the photon-pion, nucleon interaction, as will now be discussed.

A process closely related to (1.1) is

$$e^- + p \to e^- + n + \pi^+ . \tag{1.3}$$

This process has been investigated both theoretically [1,2] and experimentally [3]. It may be considered as the combination of *a*) emission by the electron of a virtual photon for which $x^2 = k_0^2 - \boldsymbol{k}^2 < 0$, and *b*) photoproduction of a pion. The electromagnetic process *a*) may be assumed to be well understood so that in this way process *b*) can be investigated for negative values of x^2 (yielding, in particular, information on the nucleon electromagnetic form factors as functions of x^2). The process (1.1) to be considered here may be looked upon as *b'*) π^- capture yielding a virtual photon which *a'*) creates an electron pair. The process *b'*) differs from *b*) only in that x^2 is now positive, ranging from 0 to μ^2, where μ is the pion mass. Thus process (1.1) is complementary to process (1.3) in the sense that the former can yield information on the electromagnetic form factors for a range of positive values of the argument x^2, whereas (1.3) relates to negative values of x^2.

Since the photons involved in (1.1) are virtual rather than real, they may have longitudinal as well as transverse polarizations. It will be shown below that separation of the longitudinal contributions can be performed rather neatly by consideration of the distribution in θ, the angle between the directions of the electron and virtual photon in the c.m. system of the electron pair (see Fig. 1). Thus process (1.1) allows a direct check on the size of the longitudinal matrix element for $\pi^- + p \to n +$ virtual γ; although such a check could also be obtained from (1.3), this has not yet been done.

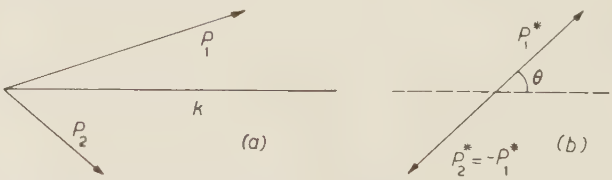

Fig. 1. – Kinematics of pair production (*a*) in the laboratory system, where $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{k} \neq 0$, $p_{10} + p_{20} = (\boldsymbol{k}^2 + x^2)^{\frac{1}{2}}$, and (*b*) in the pair c.m. system, where $\boldsymbol{p}_1^* + \boldsymbol{p}_2^* = \boldsymbol{k}^* = 0$ and $p_{10}^* + p_{20}^* = x = 2(\boldsymbol{p}_1^{*2} + m^2)^{\frac{1}{2}}$.

(1) R. H. Dalitz and D. R. Yennie: *Phys. Rev.*, **105**, 1598 (1957).
(2) S. Fubini, Y. Nambu and V. Wataghin: *Phys. Rev.*, **111**, 329 (1958).
(3) W. K. H. Panofsky, W. M. Woodward and G. B. Yodh: *Phys. Rev.*, **102**, 1392 (1956).

It is unfortunate for the measurement of these contributions (size and longitudinal) to the $\pi^-+p \to n+\text{virtual } \gamma$ matrix element that they are small, amounting to about 1% and 2%, respectively, of the total rate; on the other hand, this is advantageous for the determination of the Panofsky ratio. Since there is no longitudinal contribution to (1.2) and the size effect there is negligible, it means that the ratio of (1.1) to (1.2) is related to the Panofsky ratio by a factor determined principally by electromagnetic theory and only weakly dependent on pion physics.

Since effects of the order of one per cent are of importance to any of the above measurements, electromagnetic corrections, of relative order α, need to be considered. The determination of these corrections is the main purpose of the present paper [4].

2. – The process $\pi^-+p \to n+e^++e^-$.

Figs. 2*b*, *c*, *d*, and *e* show the diagrams which can contribute to this process in order α relative to the lowest order process, shown in Fig. 2*a*. The open circles indicate the strong-interaction vertices, the p, n, and π^- lines being omitted. It is possible for two photon lines to emerge from the strong-interaction vertex; Figs. 2*d* and 2*e* show two graphs which also contribute in relative order α. However, the probability for emission of two real photons ($\pi^-+p \to n+2\gamma$) is only 0.019α of that for the emission of one (see Appendix A), which implies that the processes of Figs. 2*d* and 2*e* contribute much less than those of Figs. 2*b* and 2*c* (which together contribute about one per cent; see below). Also, Fig. 2*a* leads to a factor $1/x^2$ favoring low-mass pairs, while Fig. 2*e* does not; thus the interference term between these two will be further reduced.

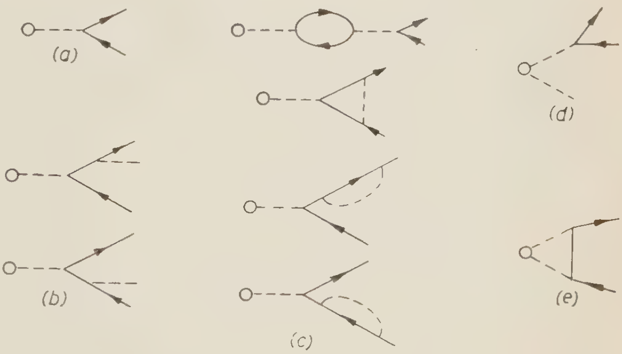

Fig. 2. – Feynman diagrams for the process $\pi^-+p \to n+e^++e^-$ $(+\gamma)$. The small circles indicate the vertex at which $\pi^-+p \to n+\text{virtual } \gamma$. Dashed lines denote photons; solid lines, electrons.

([4]) Calculations to lowest order in α for the processes (1.1) and (1.2) have been performed by R. H. DALITZ: *Proc. Phys. Soc.* (*London*), A **64**, 667 (1951), and by N. KROLL and W. WADA: *Phys. Rev.*, **98**, 1355 (1955). Size and longitudinal contributions to (1.1) have been briefly considered by R. ROCKMORE and J. TAYLOR: *Phys. Rev.*, **112**, 992 (1958).

First we must evaluate the lowest order electromagnetic contribution, Fig. 2*a*, taking account of the small corrections to the vertex $\pi^- + p \to n +$ virtual γ. Considering a system of mass $M+E$ at rest which emits a photon to become a recoiling system of mass M, KROLL and WADA [4] have obtained the following general expression [5]:

$$(2.1)\qquad \varrho_0 = \frac{\alpha}{4\pi}\int_{-\eta}^{\eta} dy \int_{2m}^{E} \frac{dx}{x}\frac{k}{\varkappa}\left[1 - \frac{x^2}{(E+M)^2 + M^2}\right]\left[\left(1 + y^2 + \frac{4m^2}{x^2}\right)R_T(x) + \right.$$
$$\left. + (1 - y^2)\frac{8(E+M)^2x^2}{(2EM + E^2 + x^2)^2} R_L(x)\right],$$

for the internal conversion coefficient (the rate of pair creation divided by the rate of photon emission), to lowest order in α. The symbols $\varkappa$ and k here denote the momenta of the real and virtual photons, respectively, and m is the electron mass. The quantity $x = (k_0^2 - \boldsymbol{k}^2)^{\frac{1}{2}}$ is the mass of the virtual photon, while $y = (p_{10} - p_{20})/|\boldsymbol{p}_1 + \boldsymbol{p}_2|$ is a measure of the energy sharing between the electron ($p_{1\mu}$) and the positron ($p_{2\mu}$). It can be shown that $y = \eta\cos\theta$, where $\eta = (1 - 4m^2/x^2)^{\frac{1}{2}}$ and θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{k}$ in the c.m. system of the electron-positron pair (see Fig. 1). The quantities R_T and R_L depend on the current $J_\mu(\boldsymbol{k})$ which produces the virtual photons:

$$(2.2)\qquad \begin{cases} R_T(x) = \sum\int d\Omega_k[|J_1(\boldsymbol{k})|^2 + |J_2(\boldsymbol{k})|^2]/\sum\int d\Omega_\varkappa[|J_1(\boldsymbol{\varkappa})|^2 + |J_2(\boldsymbol{\varkappa})|^2], \\ R_L(x) = \sum\int d\Omega_k[|J_3(\boldsymbol{k})|^2]/\sum\int d\Omega_\varkappa[|J_1(\boldsymbol{\varkappa})|^2 + |J_2(\boldsymbol{\varkappa})|^2]. \end{cases}$$

Here J_1 and J_2 are the components transverse to $\boldsymbol{k}$, and J_3 is the component in the direction of $\boldsymbol{k}$. The indicated summations are the usual sum-averages over nucleon spin states. For the present process, $M = M_n = 1839\,m$ and $E = M_p + M_\pi - M_n = 270.3\,m$, where m denotes the electron mass. It can be shown that $k/\varkappa = (1 - 0.0023\,x^2/E^2)(1 - x^2/E^2)^{\frac{1}{2}}$ to within 0.001%, so eq. (2.1) can be written as

$$(2.3)\qquad \varrho_0 = (\alpha/4\pi)\int_{-\eta}^{\eta} dy\int_{2m}^{E}(dx/x)(1 - x^2)^{\frac{1}{2}}(1 - 0.0117\,x^2)\cdot$$
$$\cdot[(1 + y^2 + 4m^2/x^2)R_T + 2.284(1 - y^2)\,x^2R_L/(1 + 0.0684\,x^2)^2] =$$
$$= (2\alpha/3\pi)\int_{2m}^{E}(dx/x)(1 - x^2)^{\frac{1}{2}}(1 - 0.0117\,x^2)\cdot$$
$$\cdot[R_T + 1.142\,x^2R_L/(1 + 0.0684\,x^2)^2](1 + 2m^2/x^2)(1 - 4m^2/x^2)^{\frac{1}{2}},$$

[5] After correction of a numerical error. Note that throughout this paper we choose units such that $\hbar = c = 1$.

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where the unit of energy has been chosen so that $E = 1$. To obtain expressions for $J_\mu(k)$ we shall use eqs. (14) and (14′) of Ref (2), making use of the fact that the matrix element for $\pi^- + p \to n + [\gamma]$ is the same as that for $[\gamma] + p \to n + \pi^+$ but for a change of sign of k_μ and q_μ. The result is,

$$(2.4)\qquad \begin{cases} |J_1(k)| = |J_2(k)| = C[e^V + k^2\mu^S]\,, \\ |J_3(k)| = C\{e^V - ek^2/(2k_0 - x^2) + [(e^V - e)/x^2 + \mu^S]\,k^2\}, \end{cases}$$

for the appropriate nucleon spin combinations, where C is a constant and some non-significant $1/M$ terms have been dropped. The quantities μ^S and e^V are given by $\mu^S = \mu'_p F_2^p + \mu'_n F_2^n + e^S(x^2)/2M$ and $e^{S,V} = e[F_1^p \pm F_1^n]$ (the « + » sign being associated with « S »), where $\mu'_{p,n}$ are the proton and neutron anomalous moments and $F_{1,2}^{p,n}$ are the corresponding form factors. Since the magnetic moment term is small, we can replace $\mu^S(x^2)$ by $\mu_0^S = \mu^S(0) = 0.87e/2M = 0.064\,e$. The vector charge can be written in the form $e^V = e(1 + r_V^2 x^2/6 + \ldots)$; electron scattering experiments indicate that $r_V = 0.80f$, or 0.57 in the present units (6,7). Eqs. (2.2) and (2.4) yield

$$(2.5)\qquad \begin{cases} R_T \doteq 1 + r_V^2/3 - 2(\mu^S/e)x^2\,, \\ R_L = 0.142\,(1 + \mu^S/e)^{-2}\{(1 - 0.466\,x^2)^{-1} + 1.88\,[r_V^2/6 + (\mu^S/e)(1 - x^2)]\}^2, \end{cases}$$

where the static approximation $k^2 \approx 1 - x^2$, $k_0 \approx 1$ has been used in the small terms. Substituting eqs. (2.5) into eq. (2.3) and integrating, one obtains

$$(2.6)\qquad \varrho_0 = \varrho_{0T0} + \varrho_{0TM} + \varrho_{0TS} + \varrho_{0L} = $$

$$= (69.00 - 0.65 + 0.55 + 1.42)\cdot 10^{-4} = 0.007\,03\,,$$

where ϱ_{0T0} is the coefficient for conversion of transverse photons without magnetic moment or size effect contributions, ϱ_{0TM} and ϱ_{0TS} are the corrections to this rate arising from the magnetic moment and size effect terms, and ϱ_{0L} is the longitudinal contribution; all are calculated to lowest order in α at the virtual photon-electron pair vertex (denoted by the first subscript 0). The corresponding distributions in x are shown in Fig. 3. The size effects have been evaluated with the experimental value $r_V = 0.80f$ (8).

(6) D. R. YENNIE, M. LÉVY and D. G. RAVENHALL: *Rev. Mod. Phys.*, **29**, 144 (1957).

(7) R. HOFSTADTER, F. BUMILLER and M. R. YEARIAN: *Rev. Mod. Phys.*, **30**, 482 (1958).

(8) It is to be noted that the transverse size effect of eq. (2.6) is about half that given by ROCKMORE and TAYLOR (4), due to an error of a factor 3 in their expressions combined with the fact that they used the earlier experimental value $r_V \doteq \frac{1}{2}\,\mu$.

To evaluate the contribution from the diagrams of Figs. 2*b* (corresponding to the emission of real radiation) and 2*c* (virtual radiative corrections), we will make use of Källén and Sabry's fourth-order vacuum polarization calculation ([9]).

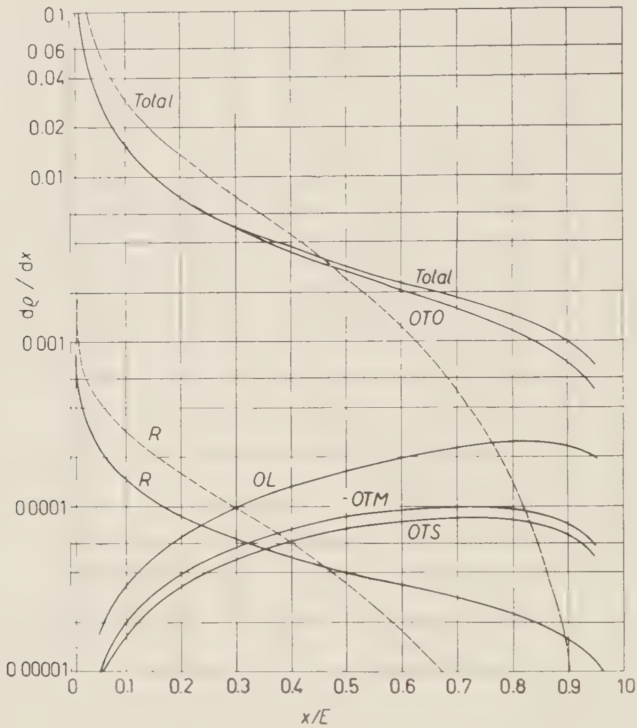

Fig. 3. – Distribution of events with respect to the electron pair mass x for the processes $\pi^0 \to \gamma + e^+ + e^-$ $(+\gamma)$ (dashed curves) and $\pi^- + p \to n + e^+ + e^-$ $(+\gamma)$ (solid curves). OTO: contribution of transverse (virtual) photons without magnetic moment and size contributions, to lowest order in α; OL: contribution of longitudinal photons, to lowest order in α; OTM, OTS: magnetic moment and size effect contributions, respectively, to the transverse rate (note sign of OTM; OTS, proportional to r_V^2, is here evaluated for $r_V = 0.80 f$); R: radiative corrections (*i.e.*, corrections of relative order α due to both real and virtual radiation). Note that all curves fall to zero at $x = 1/2m = 0.00739$; the radiative corrections reach peaks of 0.0072 (π^0) and 0.0036 (capture).

If the circles and intermediate photon lines were omitted from Figs. 2*b*, the remaining portions would correspond to $\langle 0|j^{(1)}|z\rangle$ in Källén's notation

([9]) G. Källén and A. Sabry: *Dan. Mat. Fys. Medd.*, **29**, no. 17 (1955).

(where $|z\rangle$ denotes the (e^+, e^-, γ) state), so that these contributions amount to [10]

$$(2.7)\quad \begin{cases} \varrho(2b) = D' \sum\int \mathrm{d}\Omega_k \int k^2\,\mathrm{d}k\, J^\mu(\boldsymbol{k}) J^{\nu *}(\boldsymbol{k})\, x^{-2} \sum\limits_{z:k_\lambda} \langle 0|j_\mu^{(1)}|z\rangle\langle z|j_\nu^{(1)}|0\rangle \\ \quad = D \sum\int \mathrm{d}\Omega_k \int k^2\,\mathrm{d}k\, J^\mu(\boldsymbol{k}) J^{\nu *}(\boldsymbol{k})\, x^{-2}(x^2 g_{\mu\nu} - k_\mu k_\nu)\pi_a^{(1)}(-x^2) \\ \quad = D\int (\mathrm{d}x/x)(1-x^2)^{\frac{1}{2}} \sum\int \mathrm{d}\Omega_k [|J_1(\boldsymbol{k})|^2 + |J_2(\boldsymbol{k})|^2 + x^2 |J_3(\boldsymbol{k})|^2]\pi_a^{(1)}(-x^2) \end{cases}$$

where D and D' are constants and $\sum\limits_{z:k\lambda}$ indicates summation over all electron-positron-photon states $|z\rangle$ having four-momentum k_λ; the other summation is the usual sum-average over nucleon spin states. We have made use of the approximations $k_0 \approx E - 1$ and $k \approx (1-x^2)^{\frac{1}{2}}$ (recoil effects neglected), and of $k_\mu J^\mu = 0$.

The diagrams of Fig. 2c contribute to relative order α only through their interference with those of Fig. 2a; this contribution is

$$(2.8)\quad \begin{cases} \varrho(2c) = D' \sum\int \mathrm{d}\Omega_k \int k^2\,\mathrm{d}k\, J^\mu(\boldsymbol{k}) J^{\nu *}(\boldsymbol{k})\, x^{-2} \sum\limits_{z:k_\lambda} [\langle 0|j_\mu^{(2)}|z\rangle\langle z|j_\nu^{(0)}|0\rangle + \text{c.c.}] \\ \quad = D\int (\mathrm{d}x/x)(1-x^2)^{\frac{1}{2}} \sum\int \mathrm{d}\Omega_k [|J_1(\boldsymbol{k})|^2 + |J_2(\boldsymbol{k})|^2 + x^2 |J_3(\boldsymbol{k})|^2]\pi_b^{(1)}(-x^2)\,, \end{cases}$$

where the summation $\sum\limits_{z:k\lambda}$ is here over states $|z\rangle$ containing an electron-positron pair. The constant D appearing in eqs. (2.7) and (2.8) can easily be obtained by noting that the contribution of Fig. 2a alone differs from $\varrho(2b)$ only in having $\pi_a^{(1)}(-x^2)$ replaced by $\pi^{(0)}(-x^2)$, with the latter given by [11]

$$(2.9)\qquad \pi^{(0)}(-x^2) = (\alpha/3\pi)(1+2m^2/x^2)(1-4m^2/x^2)^{\frac{1}{2}} \qquad \text{for } x^2 \geqslant 4m^2.$$

Comparison with eqs. (2.3) and (2.2) now shows that

$$(2.10)\qquad D = 2/\sum\int \mathrm{d}\Omega_k [|J_1(\boldsymbol{\varkappa})|^2 + |J_2(\boldsymbol{\varkappa})|^2]\,.$$

Thus the electromagnetic corrections to the basic process of Fig. 2a (neglecting recoil) are

$$(2.11)\qquad \varrho_R = 2\int (\mathrm{d}x/x)\,(1-x^2)^{\frac{1}{2}}(R_T + x^2 R_L)\pi^{(1)}(-x^2)\,,$$

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[10] G. KÄLLÉN: *Helv. Phys. Acta*, **25**, 417 (1952). See eqs. (43) for the definition of $\pi_a^{(1)}(-x^2)$.

[11] See ref. [9], eq. (6).

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where

$$\pi^{(1)}(-x^2) = \pi_a^{(1)}(-x^2) + \pi_b^{(1)}(-x^2) \,. \tag{2.12}$$

The function $\pi^{(1)}(-x^2)$ has been calculated by Källén and Sabry ([12]), and is plotted in Fig. 4. It will be noted that $\pi^{(1)}(-x^2)$ has a sharp peak at $x = 2m$ (at which point the e^+-e^- relative velocity vanishes); this peak arises from the

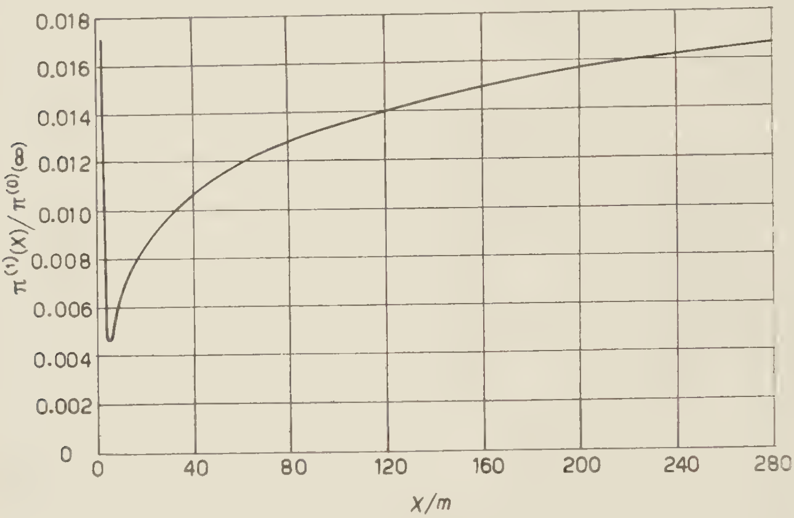

Fig. 4. – The function $\pi^{(1)}(x)$.

competition between an infinity in the matrix element and the vanishing of phase space for the pair. Such an infinity is real, and is shown in Appendix B to result from Coulomb effects (although the manner of approach to infinity at extremely small velocities is not correctly given by this order of perturbation theory). The distribution of $\varrho_R(x)$ is given in Fig. 3. Numerical integration yields

$$\varrho_R = \varrho_{RT} + \varrho_{RL} = (0.66 + 0.02)\cdot 10^{-4} \,. \tag{2.13}$$

Combination of this with eq. (2.6) yields the internal conversion coefficient, $P[\pi^- + p \to n + e^+ + e^- (+\gamma)]/P[\pi^- + p \to n + \gamma]$:

$$\varrho = \varrho_0 + \varrho_R = 0.00710 \,. \tag{2.14}$$

([12]) See ref. ([9]), eq. (49). The separate terms $\pi_a^{(1)}$ and $\pi_b^{(1)}$ are not well defined because of the infrared divergence.

3. – The process $\pi^-+p\to n+\pi^0$, $\pi^0\to\gamma+e^++e^-$.

For the process $\pi^0\to\gamma+e^++e^-$, $M=0$ and $k/\varkappa=1-x^2/\mu^2$, where μ denotes the mass of the π^0 meson ($=264.3\,m$). Eq. (2.1) yields

$$\varrho_0=\frac{4\alpha}{3\pi}\int\frac{\mathrm{d}x}{x}\left[1-\frac{x^2}{\mu^2}\right]^2\left[R_T+\frac{4R_L}{(1+x^2/\mu^2)^2}\frac{x^2}{\mu^2}\right]\left[1+\frac{2m^2}{x^2}\right]\left[1-\frac{4m^2}{x^2}\right]^{\frac{1}{2}},\tag{3.1}$$

for the contribution of lowest order in α, after doubling to allow for conversion of either photon. Invariance arguments indicate that $R_T=(k/\varkappa)(1+\varepsilon^2/x^2+...)$ where ε is a «size» associated with the intermediate states occurring in the process (the first approximation $R_T\approx k/\varkappa$ is, in fact, a result of assuming the simplest possible phenomenological form $g\varepsilon^{\mu\nu\varrho\sigma}F_{\mu\nu}F_{\varrho\sigma}$ for the π^0-2γ interaction). Since these states are assumed always to include a proton-antiproton pair, it is expected that $\varepsilon\sim 1/2M_p$; and perturbation theory indicates that it is actually smaller. Thus $\varepsilon^2x^2\leqslant(1/2M_p)^2x^2=0.005\,x^2/\mu^2$; since most events are associated with values of x *much* less than μ, it should be an excellent approximation to set $R_T=k/\varkappa$. Angular momentum conservation requires that $R_L=0$, since a real photon carries unit angular momentum about its direction of motion, while a longitudinal virtual photon carries none. Thus we find

$$\varrho_0=\frac{4\alpha}{3\pi}\int\frac{\mathrm{d}x}{x}\left[1-\frac{x^2}{\mu^2}\right]^3\left[1+\frac{2m^2}{x^2}\right]\left[1-\frac{4m^2}{x^2}\right]^{\frac{1}{2}}=0.01186\,.\tag{3.2}$$

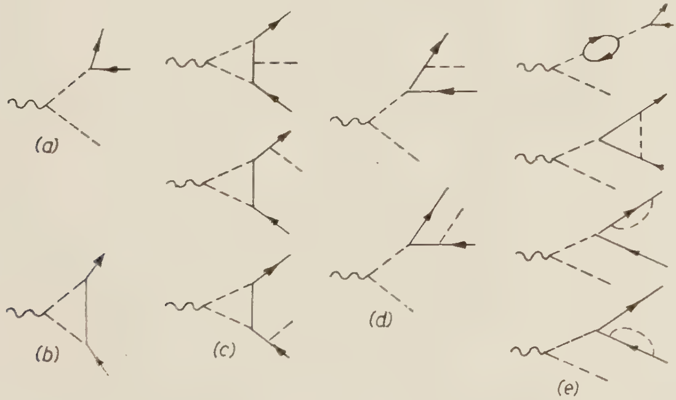

Fig. 5. – Feynman diagrams for the process $\pi^0\to\gamma+e^++e^-\,(+\gamma)$. Wavy lines denote the π^0; dashed lines, photons; and solid lines, electrons.

The above contribution corresponds to Fig. 5*a*; we will now investigate those contributions which are of order α relative to Fig. 5*a*. With the observation that the emission of a photon from the vertex producing the π^0 need

not be considered because of the low kinetic energy, and hence phase space, available, it is seen that the terms which may contribute to relative order α are those corresponding to Figs. 5*b*, *c*, *d*, and *e*. Since the *total* process here is $\pi^- + p \to n + \gamma + e^+ + e^-$, it might be thought at first that there would be interference between the processes of Figs. 2 and 5; however, the intermediate π^0 causes the (γ, e^+, e^-) system here to have combined « mass » extremely close to the π^0 mass, while for Fig. 2 the « mass » is generally much smaller.

Although Fig. 5*b* would appear to contribute to relative order α, the matrix element is actually down by an additional factor m/μ ([13]), so that its contribution is of relative order $(m/\mu)^2\alpha$ and negligible.

The graphs of Fig. 5*c* would appear to contribute to relative order α through interference with those of Fig. 5*a*. But these, too, cannot contribute appreciable since i) they must contribute negligibly for low energy of the real photon, where they reduce to Fig. 5*b*, and ii) although for large electron-pair mass (where the main contribution 5*a* is very small) they may contribute in relative order α, for small electron-pair mass they do not share the enhancement which occurs for 5*a*, *d* and *e* due to the large photon propagator.

In the radiative terms given by Fig. 5*d*, we will neglect the symmetrization with respect to the two photons, since the bremsstrahlung photon will generally have much lower energy than the other, and be emitted in a different direction. With this simplification, the contribution from 5*d* and the interference term between 5*c* and 5*a* can be evaluated (as for Figs. 2*b* and 2*c*, for π^- capture) from Ref. ([9]), yielding

$$\varrho_R = 4\int \frac{\mathrm{d}x}{x}\left[1-\frac{x^2}{\mu^2}\right]^3 \pi^{(1)}(-x^2) = 0.000105 \,. \tag{(3.2)}$$

Thus the internal conversion coefficient $P[\pi^0 \to \gamma + e^+e^-(+\gamma)]/P[\pi^0 \to \gamma + \gamma]$ is

$$\varrho = \varrho_0 + \varrho_R = 0.01196 \,. \tag{3.3}$$

The x-distributions are shown in Fig. 3.

([13]) H. Miyazawa and R. Oehme: *Phys. Rev.*, **99**, 315 (1955). This factor may be obtained as follows. This diagram must yield a contribution of the form $C(m/\mu)q\bar{u}\gamma^5 u$, where $C(m/\mu)$ is a polynomial in m/μ. The transformation $u \to \gamma^5 u$ carries this expression into its negative; while the same transformation applied to the (unintegrated) expression corresponding to the diagram has precisely the effect of inverting the sign of m. Thus $C(-m/\mu) = -C(m/\mu)$, so that C must have m/μ as a factor. Actually, there is a logarithmic divergence, but the contribution is still negligible for any reasonable size of the intermediate state of $\pi^0 \to 2\gamma$; see S. Drell: *Nuovo Cimento*, **11**, 693 (1959).

4. – Conclusions.

From eqs. (2.14) and (3.3) it follows that the Panofsky ratio is given by

$$R_p = 0.594 \frac{P(\pi^- + p \to n + \pi^0,\ \pi^0 \to \gamma + e^+ + e^-)}{P(\pi^- + p \to n + e^+ + e^-)} . \tag{4.1}$$

The coefficient 0.594 should suffer significant error neither from the approximations made above of neglecting certain diagrams and symmetrization nor from the uncertainty in the size and magnetic moment contributions. The principal uncertainty is due to our lack of knowledge of the longitudinal matrix elements and (in principle) can be removed by direct measurement of this contribution (see below). On the other hand, since the photons will not generally be observed experimentally, there is some danger of confusing events of the π^0 type with those of the capture type which have lost energy through inner bremsstrahlung; Fig. 6 indicates in a graphical way the possible extent

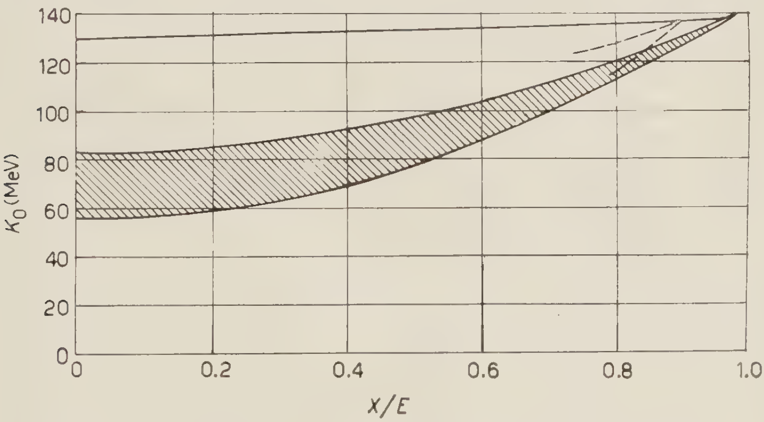

Fig. 6. – Possible combinations of mass and energy for pairs from $\pi^-+p \to n+e^++e^-$ (upper curve) and $\pi^-+p \to n+\pi^0$, $\pi^0 \to \gamma+e^++e^-$ (shaded area). Inner bremsstrahlung will move pairs slightly down and to the left; the dashed curves indicate the boundary of the region to which one point of the upper curve may be shifted in this way.

of this confusion. Fortunately, less than 0.3% of the π^0-type events have $x \geqslant 0.8$ (see Fig. 3), so that an energy uncertainty of as much as 10 MeV due to inner bremsstrahlung loss and uncertainty of measurement should still allow good separation of the two types of events.

A measurement of the contribution of longitudinal photons to process (1.1) is possible; and it would be significant, as the size of the longitudinal matrix element has never been checked, although such a check could also be obtained

from process (1.3). Separation of the longitudinal contribution is easiest if attention is restricted to events such that $x \gg 2m$, for eq. (2.1) shows that in this case the transverse and longitudinal contributions have the distributions

$$(4.2)\qquad \begin{cases} \varrho_T(y) \propto 1 + y^2 = 1 + \cos^2\theta\,, \\ \varrho_L(y) \propto 1 - y^2 = \sin^2\theta\,, \end{cases}$$

independently of x, where θ is defined in Fig. 1. The restriction $x \gg 2m$ results, in fact, in no loss of information, since the ratio $w_L(x_0)/[w(x_0)]^{\frac{1}{2}}$, where $w_L(x_0)$ and $w(x_0)$ are the cumulative distributions $w_{(L)}(x_0) = \int_{x_0}^{\mu} \mathrm{d}x\, \varrho_{(L)}(x)$ for longitudinal and total events, respectively, shows a broad maximum for x_0 between 0.5 and 0.6 (recall that $\mu \approx 1$ in the units being used, so that $2m = 0.007$). To reduce the effect of uncertainties for high-mass pairs (*e.g.*, due to the magnitude of the size effect), one might choose $x_0 = 0.4$; this would include 17 % of all events, of which 10 % are expected to be longitudinal. Thus a few thousand events of the type (1.1) should yield a rough measurement of the longitudinal contribution. The distributions (4.2) have been obtained only to lowest order in α; the question arises of whether the higher order radiative corrections (*i.e.*, those from the graphs of Figs. 2 and 5 other than 2*a* and 5*a*) may modify them appreciably. As shown in Appendix C, photon-electron interaction via a magnetic moment term does not significantly modify the distributions, for $x \gg 2m$. On the other hand, the effect of *real*-radiative corrections (inner bremsstrahlung) may warrant further investigation when accurate experiments are carried out, although it would be expected to be small.

In principle, at least, the size effect contribution can be checked by observing the distribution of pair events in x, since this distribution is dependent on the nucleon vector charge radius r_V. The magnitude of the size effect contribution as a function of x is shown in Fig. 3 for the present empirical value of $r_V = 0.80f$; it is proportional to r_V^2. Unfortunately, the effect is so small that its measurement is exceedingly difficult; the contribution to the total rate (for $r_V = 0.80f$) is only 0.8 % (cf. eq. (2.6)). This smallness is partly due to the preference for low-mass pairs, which results from the factor $1/x$ in (2.3); the contribution actually amounts to about 10 % for $x \approx \mu$. Thus if some automatic means were available for locating high-mass pairs, a measurement of the size effect might become feasible. Of course, accurate knowledge of the magnetic moment and longitudinal photon contribution would be necessary for this measurement, since these have a similar and larger effect on the x-distribution.

* * *

The author wishes to express his thanks to Professor R. H. Dalitz for suggesting this problem and for much helpful discussion concerning it.

APPENDIX A

Estimation of the matrix element for $\pi^-+p\to n+2\gamma$.

The matrix element $M_{2\gamma}$ for the process $\pi^-+p\to n+2\gamma$ is related to the matrix element M_γ for the process $\pi^-+p\to n+\gamma$ by gauge invariance. In the limit that the photon energies vanish, at least, we have

$$M_{2\gamma}=\Big[ie\sum_i \varepsilon_{2i}\sum_j \partial M_\gamma/\partial q_{ji}\Big]+[\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2] \tag{A.1}$$

where $[\boldsymbol{\epsilon}_1\leftrightarrow\boldsymbol{\epsilon}_2]$ denotes the quantity resulting from interchange of the polarization vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ in the preceding expression, and the index j distinguishes the various charged particles of momenta $\boldsymbol{q}_j$ involved in the process. Whether the expression so obtained is accurate here depends upon whether the photon energies are small compared with an appropriate characteristic mass.

The only charged particle momentum appearing in eqs. (14) and (14′) of ref. (2) is the meson momentum $\boldsymbol{q}$; since we desire the matrix element $M_{2\gamma}$ only for the case of vanishing meson momentum, only the terms of M_γ linear in $\boldsymbol{q}$ are of interest. Referring also to eqs. (13) and (13′) of ref. (2), it is seen that the terms linear in $\boldsymbol{q}$ are

$$+ie\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}\boldsymbol{q}\cdot\boldsymbol{\epsilon}_1}{k}-\frac{\mu^V}{6f^2}\left(\frac{\delta_{33}}{q^3}\right)_0[2\boldsymbol{q}\cdot\boldsymbol{k}\times\boldsymbol{\epsilon}_1+i\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1\boldsymbol{q}\cdot\boldsymbol{k}-i\boldsymbol{\sigma}\cdot\boldsymbol{k}\boldsymbol{q}\cdot\boldsymbol{\epsilon}_1]+i\mu^S\boldsymbol{\sigma}\cdot\boldsymbol{q}\times(\boldsymbol{k}\times\boldsymbol{\epsilon}_1), \tag{A.2}$$

for the case of a real photon ($k_\mu^2=0$, $\boldsymbol{k}\cdot\boldsymbol{\epsilon}=0$). The subscript on the factor $(\delta_{33}/q^3)_0$ denotes evaluation for vanishing q. The factor $(1-\omega/M)^{-1}$ which multiplies the first term of (A.2) as it appears in eqs. (14) of ref. (2) is here absorbed into the phase space, as usual; the inverse factor which would occur with the remaining terms of (A.2) need not be retained to the accuracy of ref. (2). An approximate expression for the desired matrix element now results from replacing $\boldsymbol{q}$ by $e\boldsymbol{\epsilon}_2$ and symmetrizing with respect to the two photons. When the numerical values $\mu^V=(g_p-g_n)e/2M=0.35e$, $\mu^S=(g_p+g_n)e/2M=0.064e$, $f^2=0.08$, and $(\delta_{33}/q^3)_0=0.23$ are inserted, it is found that the first term is the dominant one. But this term arose from the meson-current term of ref. (2), and must therefore correspond to the diagram in which both photons originate at one point of the meson line; so it should have the symmetrized form: $2\boldsymbol{\sigma}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2/[(\boldsymbol{k}_1+\boldsymbol{k}_2)^2+1]$. While the two expressions agree if the energy of either photon vanishes, they differ considerably for intermediate energies; this could be expected for this term, since the « characteristic mass » is evidently the meson mass, generally comparable with the photon momenta. It will be assumed that the other terms are given with sufficient accuracy by (A.1); in any case, their contribution is fairly small. Thus we find, finally,

$$\begin{aligned}M_{2\gamma}\propto e^2\{2/[(\boldsymbol{k}_1+\boldsymbol{k}_2)^2+1]+0.23\}\boldsymbol{\sigma}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2-\\ -0.23[\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_1\boldsymbol{k}_1\cdot\boldsymbol{\epsilon}_2+\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_2\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_1]+0.34i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{\epsilon}_1\times\boldsymbol{\epsilon}_2\,.\end{aligned} \tag{A.3}$$

Upon squaring this, taking the usual sum-average over nucleon polarizations, and summing over photon polarizations, one obtains

(A.4) $$\sum |M_{2\gamma}|^2 \propto e^4 \{A^2(1+c^2)[1-2(1-c)k_1k_2] + 0.93\,Ak_1k_2c(1-c^2)\}\,,$$

where $A = [1-(1-c)k_1k_2]^{-1} + 0.23$ and $c = \boldsymbol{k}_1 \cdot \boldsymbol{k}_2/k_1k_2$, to sufficient accuracy. The corresponding quantity for single-photon emission is, from eqs. (14) and (14′) of ref. (2),

(A.5) $$\sum |M_\gamma|^2 \propto 2(e+\mu^s)^2 = 2.26e^2\,,$$

so that the ratio of two-photon emission to single-photon emission is

(A.6) $$\frac{P(2\gamma)}{P(\gamma)} = \frac{1}{(2\pi)^3}\,\frac{\frac{1}{2}\int(\mathrm{d}^3k_1/2k_1)(\mathrm{d}^3k_2/2k_2)\sum|M_{2\gamma}|^2\,\delta(E-k_1-k_2)}{\int(\mathrm{d}^3k_1/2k_1)\sum|M_\gamma|^2\,\delta(E-k_1)} = 0.019\,\alpha\,.$$

for the capture by protons of negative pions at rest.

Appendix B

Electromagnetic corrections to the production of pairs having extremely low relative velocity.

The matrix element for pair production can be written in the form

(A.7) $$M = \langle e^+, e^- | \int \mathrm{d}^4r\, F_\mu(r)\,\bar\psi(r)\gamma^\mu\psi(r)\,|0\rangle\,,$$

where $\psi(r)$ is the electron field operator and $F_\mu(r)$ depends on the momenta and spins of the incoming particles. Let us consider this expression in the pair c.m. system; for small electron velocity v we then find an expression of the form:

(A.8) $$M = \int \mathrm{d}^3r'\, \tilde u_{-\boldsymbol{p}}(\boldsymbol{r}')\,\boldsymbol{\bar F}(\boldsymbol{r}')\cdot\boldsymbol{\sigma}\,u_{\boldsymbol{p}}(\boldsymbol{r}')\,,$$

where $u_{\pm p}$ are non-relativistic positron and electron spinors, respectively. Neglecting the (inessential) spin dependence and making the substitution

(A.9) $$u_{\boldsymbol{p}_1}(\boldsymbol{r}_1)\,u_{\boldsymbol{p}_2}(\boldsymbol{r}_2) = \exp\left[i(\boldsymbol{p}_1+\boldsymbol{p}_2)\cdot(\boldsymbol{r}_1+\boldsymbol{r}_2)/2\right]\varphi_{(\boldsymbol{p}_1-\boldsymbol{p}_2)/2}(\boldsymbol{r}_1-\boldsymbol{r}_2)\,,$$

we obtain

(A.10) $$M = \bar G(0)\,\varphi_{\boldsymbol{p}}(0)\,,$$

where $\bar{G}(\boldsymbol{k})$ is essentially the Fourier transform of $\bar{F}(\boldsymbol{r}')$, and depends on the pair c.m. motion. Now the lowest-order perturbation calculation corresponds to taking φ_p to be a plane wave. At extremely low velocities, the chief electromagnetic correction is due to the Coulomb interaction between the outgoing particles, and can be accounted for by using the Coulomb function $\varphi_p^C(\boldsymbol{r}_1-\boldsymbol{r}_2)$. Thus at extremely low velocities the rate of pair production W^C is related to that obtained neglecting all electromagnetic corrections by the equation

$$\text{(A.11)} \qquad W^C/W(1) = |\varphi_p^C(0)|^2/|\varphi_p^{pw}(0)|^2 = \zeta/(1-\exp[-\zeta]) ,$$

where φ^{pw} is the plane-wave function, $\zeta = 2\pi\alpha/v$, and the expression for $|\varphi_p^C(0)|^2$ is taken from HEITLER ([14]). It is interesting to compare this correction factor with that obtained from a perturbation calculation to second order in α. This latter factor is

$$\text{(A.12)} \quad W(2)/W(1) \approx \{[\pi^{(0)}+\pi^{(1)}]/\pi^{(0)}\}^2 \qquad \text{or} \qquad W(2)/W(1) = 1 + 2\pi^{(1)}/\pi^{(0)} .$$

For low v, $\eta = (1-4m^2/x^2)^{\frac{1}{2}} \to (4p_0^2-4m^2)^{\frac{1}{2}}/2p_0 = v$, where p_0 and v are evaluated in the pair c.m. system. Noting that η here is the same as Källén's δ, we find from eqs. (49), (A.7), and (6) of ref. ([9]) that

$$\text{(A.13)} \qquad \pi^{(1)} \xrightarrow[v\to 0]{} -(\alpha^2/2\pi^2)[4\varphi(-1)+2\varphi(1)+\pi^2/2] = \alpha^2/4 \quad \text{and} \quad \pi^{(0)} \xrightarrow[v\to 0]{} \alpha v/2\pi$$

Thus

$$\text{(A.14)} \qquad W(2)/W(1) \xrightarrow[v\to 0]{} 1 + \pi\alpha/v .$$

This agrees with the result of expanding (A.11) in powers of α:

$$\text{(A.15)} \qquad W^C/W(1) = 1 + \pi\alpha/v + \dots .$$

This expansion diverges for $v \leqslant \alpha$ (an extremely low velocity for present purdoses); and, in fact, the low velocity limit of (A.14) is seen to differ from the ocrrect limit

$$\text{(A.16)} \qquad W^C/W(1) \xrightarrow[v\to 0]{} 2\pi\alpha/v ,$$

resulting from (A.11), although both go to infinity as $1/v$. It should be emphasized that events with e^+-e^- relative velocity $v \leqslant \alpha$ are of no practical importance for the processes considered here; such events are an extremely small fraction of all events, and can certainly not be distinguished experimentally.

([14]) W. HEITLER: *The Quantum Theory of Radiation*, 2nd ed. (Oxford, 1944), p. 84.

APPENDIX C

Derivation of the distributions in $\cos\theta$.

These distributions, in the approximation of vanishing electron mass, may be derived very directly as follows. We shall use the representation

$$\text{(A.17)} \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the limit $m \to 0$, the Dirac equation then splits into two independent equations each involving 2-component spinors;

$$\text{(A.18)} \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{p} \mp p_0)u = 0\,.$$

The upper two components will thus correspond to a right-handed electron (mass neglected) or a left-handed positron; and the lower two components will correspond to a left-handed electron or a right-handed positron. As before, let J_μ denote the current due to the meson capture process; then the pair-production matrix element is proportional to $e\bar{\psi}J_\mu\gamma^\mu\psi$, where the usual first order term

$$\text{(A.19)} \qquad H_1 = eA_\mu\bar{\psi}\gamma^\mu\psi\,,$$

has been used for the interaction between the photon and electron fields. If we consider the process in the c.m. system of the pair, where J_0 vanishes by the current conservation condition $k_\mu J_\mu = 0$, and use the representation (A.17), the matrix element becomes $au^\dagger\boldsymbol{J}\cdot\boldsymbol{\sigma}v$, where u and v are either both « upper » or both « lower » two-component spinors and a is a constant. Now let us introduce coordinates ξ, η and ζ such that the ζ-axis is in the direction of the electron momentum, and use the standard representation for $\boldsymbol{\sigma}$, with σ_ζ diagonal (so that, *e.g.*, the top component of the four corresponds to a right-handed electron, while the bottom component corresponds to a left-handed electron). The matrix element now becomes $a(J_\xi \pm J_\eta)$, since σ_ξ connects *electrons* only with *electrons*. Thus the longitudinal component of the current, J_3 (*i.e.*, the one in the direction of $\boldsymbol{k}$) gives rise to a matrix element $aJ_3\sin\theta\cdot(\cos\varphi + i\sin\varphi)$ (where, of course, θ is the angle between the electron and photon directions). The transverse contribution may be evaluated by choosing the x_1-axis to lie along the ξ-axis, whereupon that matrix element becomes $a(J_1 \pm iJ_2\cos\theta)$. The corresponding transition probabilities are thus

$$\text{(A.20)} \qquad W_T \propto |J_1|^2(1+\cos^2\theta) \qquad \text{and} \qquad W_L \propto |J_3|^2\sin^2\theta\,,$$

since $|J_2| = |J_1|$.

It can now easily be seen that although the form (A.20) taken by these distributions is modified by the anomalous magnetic moment of the electron, the effect is negligible. Thus, let us assume an interaction of the form

$$H_2 = eA_\mu k_\nu \overline{\psi}\sigma^{\mu\nu}\psi \, . \tag{A.21}$$

Proceeding as before, the matrix element is again found to be $u^\dagger \boldsymbol{J}\cdot\boldsymbol{\sigma}v$, but with one of u and v being an « upper » spinor, and the other, a « lower » spinor. In this case it is only σ_ξ that connects electrons with positrons so that production of pairs occurs by matrix elements proportional to $\pm J_\xi$. The angular distributions are now found to be

$$W'_T \propto |J_1|^2 \sin^2\theta \qquad \text{and} \qquad W'_L \propto |J_3|^2 \cos^2\theta \, . \tag{A.22}$$

Since the two interactions connect different sets of states (« corresponding » two-component spinors for the first and « opposite » ones for the second), there is no interference term (in the limit $m \to 0$). Thus, since the magnetic moment interaction resulting from electromagnetic theory is of order $\varkappa H_2$, magnetic moment effects will modify the $y \doteq \cos\theta$ distributions only to order $\varkappa^2$, provided $p_0 \gg m$ (or, equivalently, provided $x = 2p_0 \gg m$).

RIASSUNTO (*)

Si calcolano le correzioni elettromagnetiche per i processi $\pi^-+p \to n+e^++e^-$ e $\pi^0 \to \gamma+e^++e^-$. Queste correzioni dipendono dall'energia nel sistema del c.m. della coppia. Nel caso di bassa energia della coppia nel sistema del centro di massa, le correzioni sono relativamente grandi e dovute principalmente all'attrazione columbiana elettrone-positrone; esse decrescono rapidamente ad un minimo del 0.5% con l'energia del c.m. crescente e poi aumentano lentamente al valore dell'1.7% per il massimo valore dell'energia del c.m. Per i due processi le correzioni al tasso totale ammontano a circa 1%. In termini di questi tassi, il rapporto di Panofsky risulta

$$P_P = 0.594 P[\pi^-+p \to n+\pi^0,\ \pi^0 \to \gamma+e^++e^-]/P[\pi^-+p \to n+e^++e^-] \, .$$

Per il processo $\pi^-+p \to n+e^++e^-$, si discute il problema di identificare empiricamente i contributi dovuti all'interazione tramite la componente longitudinale del campo elettromagnetico e degli effetti di dimensione. Si prevede che tali contributi ammontino solo al 2% e al 0.8% rispettivamente. Si discute brevemente anche il processo $\pi^-+p \to n+2\gamma$.

(*) *Traduzione a cura della Redazione.*

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^{60}Ni (γ-γ) Angular Correlation.

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(ricevuto il 2 Marzo 1960)

Summary. — The ^{60}Ni (γ-γ) angular correlation has been remeasured using time-of-flight techniques. The results obtained are in close agreement with the theoretical correlation function for the γ-γ cascade transition: $4(E2)2(E2)0$. The merits of the technique employed are discussed.

1. – Introduction.

The importance of measurements of angular distributions and correlations in nuclear reactions has become increasingly evident in recent years. In the particular case of γ-γ angular correlations, especially where the anisotropy is of small magnitude, the experimental measurements have been fraught with difficulties which in some cases have led to errors which have cast doubt on the correct theoretical interpretation of the correlation. In order to remove these errors it has been found necessary to make certain important corrections to the experimental data. These corrections usually include:

1) a correction for the finite angular resolution of the detectors;

2) a correction for scattering between the two counters;

3) a correction for uncertainties in the subtraction of the background level;

4) the conventional inclusion of the standard errors due to the finite number of coincidence counts.

If we consider first the last factor, it is evident that in order to obtain accurate results at all it is necessary to minimize the purely statistical error associated with the total number of coincidences taken at any one angular setting. In any practical experiment it is necessary to decide on the acceptable statistical error, which in turn fixes the total number of coincidence counts

to be taken on the average at each angle. Having fixed this quantity, the interrelation between the factors on which the total coincidence count depends is also fixed. These factors are the γ-ray source intensity, the solid angles and detection efficiencies of the detectors, the correlation function and the length of time necessary for each measurement.

In any γ-γ correlation experiment the rate of true coincidences N_t is given by

$$N_t = N_0 \Omega_1 \eta_1 \Omega_2 \eta_2 \cdot W(\theta)\,,$$

where N_0 is the source intensity, η_1 and η_2 are the detection efficiencies of the two detectors, Ω_1 and Ω_2 are the solid angles which the detectors subtend at the source, and $W(\theta)$ is the correlation function.

On the other hand the random coincidence rate N_r is given by

$$N_r = N_0^2 \Omega_1 \eta_1 \Omega_2 \eta_2 \cdot 2\tau\,,$$

where τ is the resolving time of the coincidence circuit used. Hence the ratio of the true to the random coincidence rate is

$$\frac{N_t}{N_r} = \frac{W(\theta)}{2N_0\tau}\,.$$

It is clear from this relation that, having chosen the minimum acceptable value of the ratio N_t/N_r in order to secure a given accuracy, the product $N_0\tau$ is fixed. However, the same ratio of N_t/N_r may be obtained with a higher source strength if the resolving time τ can be made smaller. The advantages of a greater source strength are that, for a given counting rate, the detectors may be placed at a larger distance from the source, thereby reducing both the solid angles (and hence the corrections for solid angle) and the scattering of γ-rays between the two counters. The importance of this is the more apparent if we recall that, in order to minimize the factors leading to the necessity of corrections 2) and 3), it has become customary to resort to inconveniently heavy lead shielding of the detectors.

2. – The ^{60}Ni angular correlation.

The ^{60}Ni angular correlation has been extensively studied, especially by DEUTSCH and his collaborators (1). It has been shown that the γ-γ correlation is consistent with the assignment $4(E2)2(E2)0$. However, AEPPLI *et al.* (2)

(1) E. L. BRADY and M. DEUTSCH: *Phys. Rev.*, **78**, 558 (1950).
(2) H. AEPPLI, H. FRAUENFELDER and M. WALTER: *Helv. Phys. Acta*, **24**, 335 (1951).

found the anisotropy to be somewhat too small to be compatible with this assignment, and some doubt was cast on its validity. The correlation was further studied by KLEMA and McGOWAN ([3]) and by LAWSON *et al.* ([4]), using improved methods. It is noteworthy that in both cases the uncorrected experimental data gave values of anisotropy lower than the theoretical prediction, but that they were brought into agreement with theory when the necessary corrections were made. It seems clear that the neglect of proper consideration of these corrections also explains the discrepancy found by AEPPLI *et al.*

A discussion of the necessary corrections to measurements of γ-γ angular correlations as they are ordinarily performed has been given by ROSE ([5]) and others ([6]). Their proper estimation is laborious and leads to uncertainties in the final result. There appear also to be other factors involved which lead to uncertainties which require for their elimination tedious calibration experiments. An example is the observation by LAWSON *et al.* ([4]) that the measured directional correlation depended strongly on the settings of the pulse height discriminators in the detecting system. This was ascribed to a variation in the effective solid angle of the detectors which depended on the bias level.

Since the ^{60}Ni angular correlation has assumed almost the status of a standard correlation for checking the performance of apparatus designed to measure other angular correlations, it seemed desirable to remeasure the correlation with an apparatus free from most of the disadvantages of conventional techniques, and for which the corrections mentioned above would be comparatively unimportant. For this puspose we made use of an associated particle time-of-flight spectrometer which was built primarily for use in neutron spectroscopy ([7]).

3. – Experimental arrangement.

A schematic diagram of the electronic apparatus, which incorporates a time-to-pulse height converter covering a range of 100 ns and a conventional fast-slow system, is shown in Fig. 1. The ^{60}Co source used was a pile irradiated cobalt cylinder of dimensions 2 mm × 2 mm and specific activity of one millicurie (*). The source was placed at the centre of a turntable,

([3]) E. D. KLEMA and F. K. McGOWAN: *Phys. Rev.*, **87**, 524 (1952) and **91**, 616 (1953).
([4]) J. S. LAWSON and H. FRAUENFELDER: *Phys. Rev.*, **91**, 649 (1953).
([5]) M. E. ROSE: *Phys. Rev.*, **91**, 610 (1953).
([6]) M. WALTER, O. HUBER and W. ZIIUTI: *Helv. Phys. Acta*, **23**, 697 (1950).
([7]) J. B. GARG: *Nucl. Instr.*, **6**, 72 (1960).
(*) Supplied by the Radiochemical Centre, Amersham, Berks., England.

the two γ detectors being placed on arms capable of rotation, each at a distance of 35 cm from the source. The counters consisted of NE102 plastic scintillators, 2 in. diameter $\times$ 2 in. long, mounted on RCA 6810A photomul-

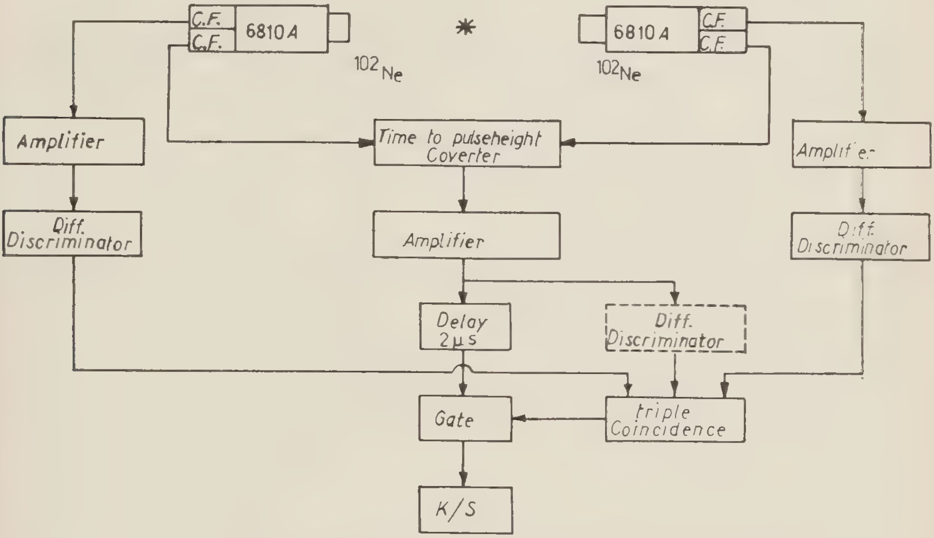

Fig. 1. – Schematic diagram of the electronic apparatus.

tipliers. Hence the half-angle subtended at the source by the detectors was $\simeq 4°$, whilst the solid angle was $\simeq 0.005$ steradians. Since no γ-rays other than those from the cascade of interest are emitted from the source, it was not necessary to make any energy selection. The discriminator biases in the slow side channels were adjusted to accept all γ-rays of energy greater than 250 keV; with this setting the detection efficiency of the detectors for 1 MeV γ-rays was about 20 %. A coincidence rate of about 2 counts per second was obtained. It may be apposite at this point to reiterate that it is because the coincidence resolving time of the instrument is $\leqslant$ 1 nanosecond that it is possible to achieve a good ratio of true to random counts whilst using a source of high strength. The use of a strong source allows one to minimize the solid angle correction and to reduce the scattering between detectors by increasing the distance between them and the source. Since both the true coincidence and random spectra are simultaneously displayed on a kicksorter, the estimation of the random background is simple and certain, especially since it is quite uniform.

The coincidence spectra for the angular settings of 90°, 120°, 150° and 180° are shown in Fig. 2. It is clear from the undistorted true coincidence peak in conjunction with the uniform random background that, although no lead shielding at all was used, the scattering between counters was negligible. If

scattered γ-rays had been important, the effect would have been evident ni the spectrum since they would occur in a different channel number from that of the main γ-γ peak, due to their different time of flight.

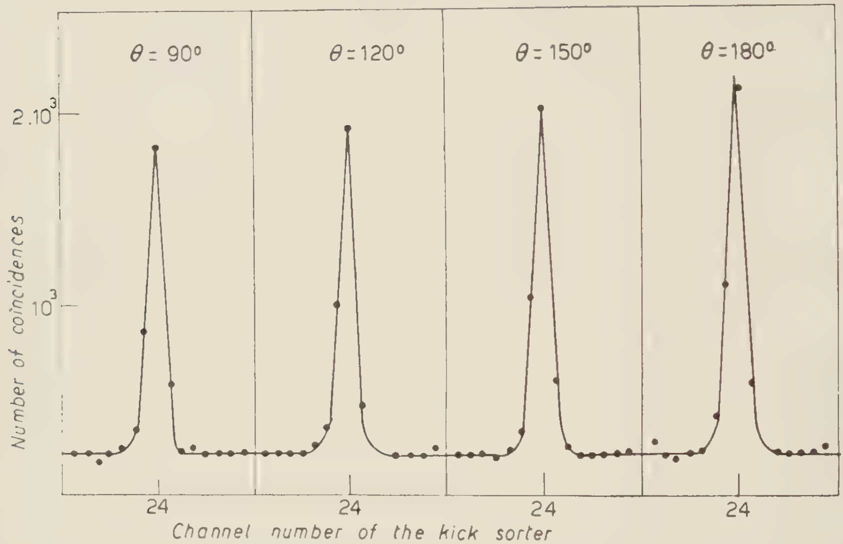

Fig. 2. (γ-γ) coincidence spectra at different angular positions of movable counter.

Apart from the unconventional technique, the angular correlation was measured in the usual way. Fig. 3 shows the experimental results, compared with the theoretical correlation curve for the $4(E2)2(E2)0$ assignment. No correc tion was necessary for scattering or background uncertainties, and the solid angle correction is very small.

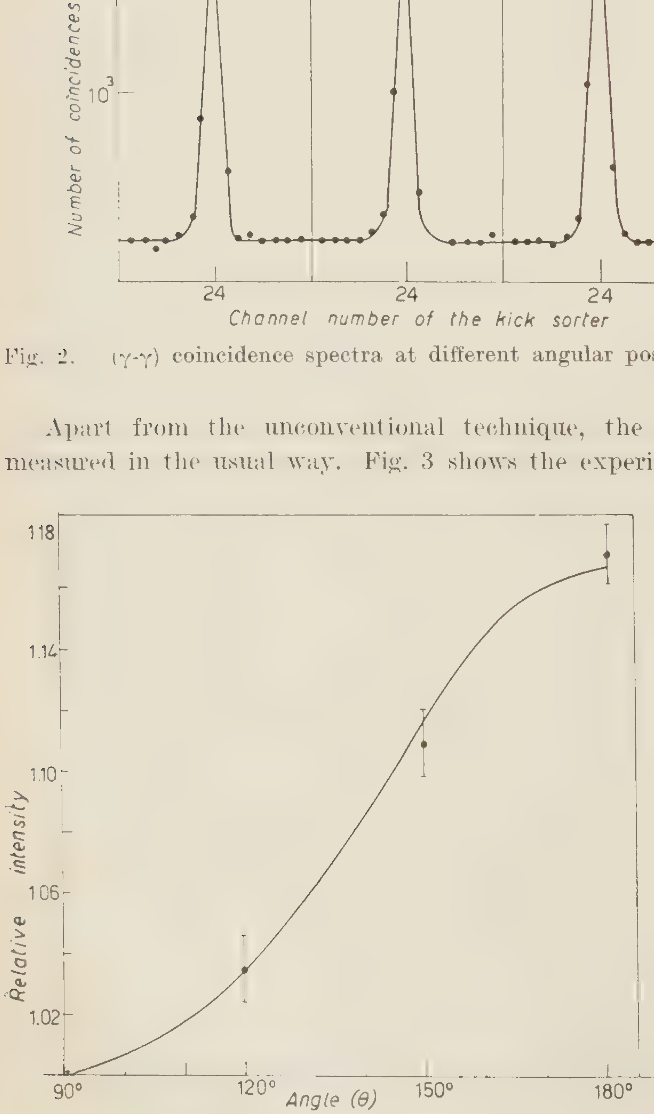

Fig. 3. – ^{60}Ni (γ-γ) angular correlation function. The solid curve is the theoretical curve $W(\theta) = 1 + 0.102\,P_2(\cos\theta) + 0.009\,P_4(\cos\theta)$.

The anisotropy, calculated as

$$A = \frac{W(180°) - W(90°)}{W(90°)},$$

and taking the mean of twelve measurements, is 0.171 ± 0.01 which is in good agreement with the theoretical value of 0.1667.

4. – Conclusion.

The technique presented in this paper is limited in its present form, using organic scintillators, to the study of simple cascades. In order to apply the technique to the study of complex cascades, where energy selection is necessary, the inorganic scintillator NaI(Tl) must be used. It is difficult to obtain very short resolving times with this fluor due to its long decay time, although BEGHIAN *et al.* ([8]) have recently obtained a time resolution of 2.7 nanoseconds by employing cooled unactivated NaI. We have obtained a resolution of about 3 ns using a NaI(Tl) fluor in one side and an NE102 fluor in the other side of a fast coincidence system. It is to be expected that still better time resolution will be obtained in the future, whilst retaining the energy resolution characteristic of NaI fluors. It is hoped that the technique presented here will then constitute a valuable tool in the study of complicated cascade processes.

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One of us (J.B.G.) would like to gratefully acknowledge the tenure of a Turner and Newall Post-Doctoral fellowship at the University of Manchester.

([8]) L. E. BEGHIAN, G. H. R. KEGEL and R. P. SCHARENBERG: *Rev. Sci. Instr.*, **29**, 753 (1958).

RIASSUNTO (*)

Si è misurata nuovamente la correlazione angolare ^{60}Ni(γ-γ), usando le tecnica del tempo di volo. I risultati ottenuti sono in ottimo accordo con la funzione teorica di correlazione per la transizione a cascata γ-γ: $4(E2)2(E2)0$. Si discutono i vantaggi della tecnica impiegata.

(*) *Traduzione a cura della Redazione.*

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On the $\mathcal{N}$-Component in Extensive Air Showers.

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(ricevuto il 18 Marzo 1960)

Summary. — According to Nikolskij *et al.* (1957), the number of nuclear-active ($\mathcal{N}$) particles in air showers increases fairly slowly, as $E^{0.2}$, for showers ranging in energy from 10^{13} eV to 10^{15} eV. At energies $>10^{15}$ eV the increase is more rapid and is $\propto E$. This variation in the $\mathcal{N}$-component with the size of the shower and certain other observed features of air showers have been interpreted previously as an indication of a change in the nature of nuclear interactions at very high energies or as an indication of specific details in the nature and composition of the primary cosmic radiation, such as the existence of a cut-off in the magnetic rigidity of the particles and the predominance of heavy primaries at very high energies. In this paper an attempt is made to explain the above observed features in terms of certain characteristics of nucleon-nucleus (n-$\mathcal{N}$) pion-nucleus (π-$\mathcal{N}$) collisions; the term «nucleus» here refers to an «air nucleus». The main aspect of the (Nikolskij) curve, for the abundance of $\mathcal{N}$-particles *vs.* shower energy, considered here, is not the rapid increase of the $\mathcal{N}$-component in showers of energy $>10^{15}$ eV, nor the existence of a change in the abundance of $\mathcal{N}$-particles in going from small showers to big showers, but rather the fairly slow increase in the number of $\mathcal{N}$-particles in showers of energy $(10^{13}\div10^{15})$ eV. This slow increase, which is contrary to general experience in cascade calculations, puts stringent limitations on the choice of parameters that characterize n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions. It is shown that n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions are largely elastic and that the number of secondary nucleons produced in such collisions is quite small (of the order of one or two); further, in a π-$\mathcal{N}$ collision, the energy going into the nucleon component is a very small fraction of the energy of the incident pion. These features also result in a rapid increase in the $\mathcal{N}$-component of big showers; this arises as a result of the complicated role played by charged pions owing to their finite life time. In small showers the charged pions have only small energies and hence they mostly decay and do not contribute appreciably

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to the $\mathcal{N}$-component. In big showers, however, the pions have sufficiently high energy to make nuclear interactions before decaying and owing to the large multiplicity of pions produced in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions, there is a rapid increase in the number of $\mathcal{N}$-particles with increasing energy of the shower. Some experiments are suggested to distinguish between the various possible explanations.

1. – Introduction.

NIKOLSKIJ *et al.* ([1]) have reported observations on the nuclear active ($\mathcal{N}$) component of extensive air showers recorded at an altitude of 3 860 m. According to them, the number of $\mathcal{N}$-particles in a shower increases fairly slowly for showers ranging in energy from 10^{13} eV to 10^{15} eV and varies roughly as $E^{0.2}$ over this interval. At energies $> 10^{15}$ eV the increase is more rapid and varies as E itself (see Fig. 3). A somewhat similar behavior has been observed by VAVILOV and NIKOLSKIJ ([2]) in the total number of penetrating particles, (mostly, μ-mesons), when this is plotted as a function of the primary energy.

NIKOSLKIJ *et al.* have suggested that the sudden increase in the abundance of the $\mathcal{N}$-component, and in the total number of penetrating particles, in going from small to very large showers can be interpreted in terms of a change in the character of the elementary act of nuclear collision. Alternatively, it has been suggested by PETERS ([3]) that the above phenomena can be explained in terms of specific details of the charge and energy spectrum of the primary cosmic radiation. According to this view there could be a high energy cut-off in the primary spectrum when this is expressed in terms of the momentnm per nucleon of the primary particle. A rigidity cut-off of this nature could be related to the acceleration mechanisms or to the strengths of the magnetic fields needed to contain the particles. If the cut-off is placed at a rigidity corresponding to an energy of about 10^{15} eV for protons, then showers which have energies greater than 10^{15} eV must be due to heavy primary nuclei which possess these energies as total energies while the energy per nucleon is still below the cut-off value. In such showers, which are a composite of all the showers caused by the constituent nucleons, the $\mathcal{N}$-component would increase as fast as the energy of the shower. The hypothesis in this form is faced with a difficulty, namely, the experimentally established existence of showers of very high energy $\sim 10^{18}$ eV. To explain their presence it is suggested that

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([1]) S. I. NIKOLSKIJ, N. VAVILOV and V. V. BATOV: *Sov. Phys. Dokl.*, **1**, 625 (1957).

([2]) N. VAVILOV and S. I. NIKOLSKIJ: *Soviet Physics - Journ. Exp. Teor. Phys.*, **5**, 1078 (1957).

([3]) B. PETERS: private communication.

other sources of cosmic radiation contribute to a much weaker extent to the observed primary radiation; the radiation due to these sources may be subject to a much higher rigidity cut-off.

In the above attempts the main aim has been to explain the existence of a change in the abundance of $\mathcal{N}$-particles at an energy of about 10^{15} eV and the subsequent rapid increase in the number of $\mathcal{N}$-particles. No explanation has been offered for the rather slow rise in the number of $\mathcal{N}$-particle as $E^{0,2}$ in the energy interval from 10^{13} eV to 10^{15} eV. This slow rise is contrary to general experience in cascade calculations where one finds that the number of particles in a cascade increases as fast or faster than the energy of the primary initiating the cascade. One finds this, for example, in the specific cases worked out by UEDA and OGITA (4) and FUKUDA *et al.* (5).

In trying to understand this feature, it was realized that the observed slow increase in the $\mathcal{N}$-component in small showers puts rather stringent limitations on the choice of parameters that characterize nucleon-nucleus (n-$\mathcal{N}$) and pion nucleus (π-$\mathcal{N}$) collisions. As a result of these limitations it is deduced that n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions are largely elastic and that the number of secondary nucleons produced in such collisions is quite small, (of the order of one or two); further, in a π-$\mathcal{N}$ collision, the energy going into the nucleon component is a very small fraction of the energy of the incident pion. These characteristics which have been fixed in order to get a slow increase in the number of $\mathcal{N}$-particles in small showers lead, however, to a rapid increase of the $\mathcal{N}$-component of big showers. This arises as a result of the complicated role played by charged pions owing to their finite life-time. In small showers the charged pions have only small energies and hence they mostly decay and do not contribute appreciably to the $\mathcal{N}$-component. In big showers, however, the charged pions have sufficiently high energies to make nuclear interactions before decaying and owing to the large multiplicity of pions produced in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions, there is a rapid increase in the number of $\mathcal{N}$-particles with increasing energy of the shower. Thus the results obtained in this paper explain fairly satisfactorily, in terms of certain characteristics of n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions, the observed variation in the number of $\mathcal{N}$-particles in the entire energy range from 10^{13} eV to 10^{17} eV.

One could maintain that, whilst this explanation for the slow increase of the $\mathcal{N}$-component in small showers is correct, it is possible that the observed change in the abundance of $\mathcal{N}$-particles at an energy $\sim 10^{15}$ eV may be caused by a change in the nature of the elementary act of nuclear interaction or as a result of the role played by heavy primary nuclei. These possibilities are taken up in the discussion at the end of this paper.

(4) A. UEDA and N. OGITA: *Progr. Theor. Phys.*, **18**, 269 (1957).
(5) H. FUKUDA, A. UEDA and N. OGITA: *Progr. Theor. Phys.*, **21**, 29 (1959).

2. – Method of calculation.

We first divide the $\mathcal{N}$-component observed at a given depth $\boldsymbol{x}$ into two groups as follows: the first group of particles, designated as $N_0(E_0, E, \boldsymbol{x})$ are those nucleons which when traced back to the primary particle which initiated the shower do not involve a pion link anywhere; all other $\mathcal{N}$-particles, designated as $\bar{N}(E_0, E, \boldsymbol{x})$, are either pions or nucleons which are secondary to pions. The rate of increase of particles in the group N_0 with the size of the shower, will depend only on the characteristics of the nucleon-nucleus collision but for $\bar{N}$ this will also depend on the characteristics of the pion-nucleus collision. We find, as discussed below, that whilst one can get a fairly slow increase in the number of particles in the group N_0 by a suitable choice of the parameters describing the n-$\mathcal{N}$ collision, the increase in the number of particles in the group $\bar{N}$ is always as fast or faster than the increase in the energy of the shower, if we take reasonable values for the multiplicities of pions produced in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions. Hence, to fit the calculated number of $\mathcal{N}$-particles with the number experimentally observed, we first choose the parameters of the n-$\mathcal{N}$ collision in such a way that the particles of the group N_0 approximately reproduce the experimental curve of Nikolskij *et al.*, for energies below 10^{15} eV. We then choose the parameters of the π-$\mathcal{N}$ collision so that the particles of the second group $\bar{N}$ make only a relatively small contribution at these energies ($< 10^{15}$ eV); then the rate of increase of the $\mathcal{N}$-component in small showers is determined essentially by the term N_0.

The diffusion equations for the nucleons and charged pions are given by

$$(1)\qquad \frac{\partial N(E, E_0; \boldsymbol{x})}{\partial \boldsymbol{x}} = -N(E, E_0; \boldsymbol{x}) + \int_E^{E_0} f_{\mathcal{N}\mathcal{N}}(E, E')\, N(E', E_0; \boldsymbol{x})\, \mathrm{d}E' + \int_E^{E_0} f_{\mathcal{N}\pi}(E, E')\, \Pi(E', E_0; \boldsymbol{x})\, \mathrm{d}E',$$

$$(2)\qquad \frac{\partial \Pi(E, E_0; \boldsymbol{x})}{\partial \boldsymbol{x}} = -\left(1 + \frac{B}{E\boldsymbol{x}}\right) \Pi(E, E_0; \boldsymbol{x}) + \int_E^{E_0} f_{\pi\mathcal{N}}(E, E')\, N(E', E_0; \boldsymbol{x})\, \mathrm{d}E' + \int_E^{E_0} f_{\pi\pi}(E, E')\, \Pi(E', E_0; \boldsymbol{x})\, \mathrm{d}E'.$$

Here $N(E, E_0, \boldsymbol{x})\,\mathrm{d}E$ and $\Pi(E, E_0, \boldsymbol{x})\,\mathrm{d}E$ represent the differential numbers of nucleons and charged pions having energies between E and $E+\mathrm{d}E$; E_0 is the

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energy of the primary nucleon initiating the shower. The symbol x stands for the atmospheric depth measured in terms of the collision unit which is taken to be $70\ \text{g cm}^{-2}$. Equations (1) and (2) are to be solved subject to the boundary condition

$$N(E, E_0; 0) = \delta(E - E_0)\,; \qquad \Pi(E, E_0; 0) = 0\,. \tag{3}$$

$f_{\pi\mathcal{N}}(E, E')$ represents the production spectrum of charged pions when a nucleon of energy E' collides and so on; B is a constant of the order $1.4\cdot 10^{11}$ eV. The solution of equations (1) and (2) by the method of successive generations can be written as

$$N(E, E_0; x) = \sum_{l=0}^{\infty} \exp[-x]\,\frac{x^l}{l!}\, n_l(E, E_0)\,, \tag{4}$$

$$\Pi(E, E_0; x) = \sum_{l=0}^{\infty} \exp[-x]\,\frac{x^l}{l!}\, \pi_l(E, E_0)\,, \tag{5}$$

n_l and π_l satisfy the recurrence relations:

$$n_l(E, E_0) = \int_E^{E_0} f_{\mathcal{N}\mathcal{N}}(E, E')\, n_{l-1}(E', E_0)\,\mathrm{d}E + \int_E^{E_0} f_{\mathcal{N}\pi}(E, E')\,\pi_{l-1}(E', E_0)\,\mathrm{d}E'\,, \tag{6}$$

$$\pi_l(E, E_0) = \frac{1}{1 + B/lE}\left\{\int_E^{E_0} f_{\pi\mathcal{N}}(E, E')\, n_{l-1}(E', E_0)\,\mathrm{d}E' + \int_E^{E_0} f_{\pi\pi}(E, E')\,\pi_{l-1}(E', E_0)\,\mathrm{d}E'\right\}, \tag{7}$$

and the initial conditions $n_0(E, E_0) = \delta(E - E_0)$; $\pi_0(E, E_0) = 0$. We give now another semi-analytical solution of the equations (1) and (2) which is the basis of our calculations. If $N_0(E, E_0; x)$, as defined earlier, is the solution of equation (1) when the last term on the R.H.S. of equation (1) is neglected and $N_0(E, E_0; x)$ satisfies the boundary condition, $N_0(E, E_0; 0) = \delta(E - E_0)$, then we have the following solutions of equations (1) and (2):

$$N(E, E_0; x) = \sum_{l=0}^{\infty} N_l(E, E_0; x) = N_0(E, E_0; x) + \sum_{l=1}^{\infty} \int_0^x \mathrm{d}y\, \exp[-y]\,\frac{y^l}{l!} \int_E^{E_0} \mathrm{d}E'\, N_0(E, E'; x - y) \int_{E'}^{E_0} \mathrm{d}E''\, f_{\mathcal{N}\pi}(E', E'')\,\pi_l(E'', E_0)\,, \tag{8}$$

$$\Pi(E, E_0; x) = \exp[-x]\, x^{-B/E} \int_0^x y^{B/E} \exp[y]\,\mathrm{d}y \int_E^{E_0} f_{\pi\mathcal{N}}(E, E')\Big\{\sum_{l=0}^{\infty} N_l(E', E_0; y)\Big\}\mathrm{d}E' + \sum_{l=1}^{\infty} \exp[-x]\,\frac{x^l}{l!(1 + B/lE)} \int_E^{E_0} f_{\pi\pi}(E, E')\,\pi_{l-1}(E', E_0)\,\mathrm{d}E'\,, \tag{9}$$

where π_l is the same quantity defined in equation (5).

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To investigate the rate of increase of $N_0(E, E_0, \boldsymbol{x})$ with E_0, we take a simple production spectrum of secondary nucleons produced in n-$\mathcal{N}$ collisions, namely,

$$f_{\mathcal{N}\mathcal{N}}(E, E') = n\,\delta\left(\frac{\gamma}{n}E' - E\right), \tag{10}$$

which implies that we get « n » secondary nucleons sharing equally a fraction γ of the incident energy. Then $N_0(E, E_0, \boldsymbol{x})$, (integral number), can be expressed as an integral in the complex plane:

$$N_0(E, RE_0; \boldsymbol{x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}s}{s}\left(\frac{E_0}{E}\right)^s \exp\left[-\nu(s)\boldsymbol{x}\right], \tag{11}$$

and an approximate value of it is given in a parametric form by the saddle point method as follows:

$$N_0(E, E_0; \boldsymbol{x}) \simeq \frac{(E_0/E)^{s_0}\exp\left[-\nu(s_0)\boldsymbol{x}\right]}{s_0\sqrt{2\pi[\nu''(s_0)\boldsymbol{x} + 1/s_0^2]}}, \tag{12}$$

and

$$\log E_0/E = \nu'(s_0)\boldsymbol{x} + \frac{1}{s_0}, \tag{13}$$

where $\nu(s) = 1 - n\exp[-\alpha s]$ and $\alpha = \log n/\gamma$. From (12) and (13) we get for the rate of increase of nucleons

$$\frac{\mathrm{d}\log N_0(E, E_0; \boldsymbol{x})}{\mathrm{d}\log E_0} = \tag{14}$$

$$= \frac{n\alpha^2 s_0^3\boldsymbol{x}\exp[-\alpha s_0]}{1 + n\alpha^2 s_0^2\boldsymbol{x}\exp[-\alpha s_0]}\left[1 + \frac{s_0(1-\alpha s_0/2)}{1 + n\alpha^2 s_0^2\boldsymbol{x}\exp[-\alpha s_0]}\right] \simeq s_0\,\frac{A}{1+A},$$

where $A = n\alpha^3 s_0^2\boldsymbol{x}\exp[-\alpha s_0]$.

3. – Choice of the parameters and the spectra.

We immediately see that if $n = \gamma = 1$, then $\alpha = 0$, and the rate of increase of nucleons is zero which is obvious since $n = \gamma = 1$ implies that there is always only one nucleon at a given depth whatever be the energy of the primary. In Fig. 1 we have plotted $N_0(E, E_0, \boldsymbol{x})$ as a function of E_0 for various values of n and γ, taking $\boldsymbol{x} = 10$. We have also reproduced the lower part of the Nikolskij curve. From the curves given there it is clear that the absolute numbers of particles are very sensitive to n and the slopes of the curves

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are sensitive to γ. It is obvious from the expression (14) or the curves in Fig. 1 that in order to get a slow rise in number $N_0(E_0)$ with E_0, we must have a small value of n (about 2 or less) and γ should tend towards unity. The choice of γ as small as 0.5 is definitely ruled out since $N_0(E_0)$ for this choice increases as fast as E_0. Remembering that there is still a contribution $\bar{N}(E, E_0, x)$ to be added on to get the total $\mathcal{N}$-component and that this would cause the slope of the curve to increase, we take the values $\gamma = 0.9$ and $n = 1.5$ as a suitable choice to obtain the best fit with the experimental curve.

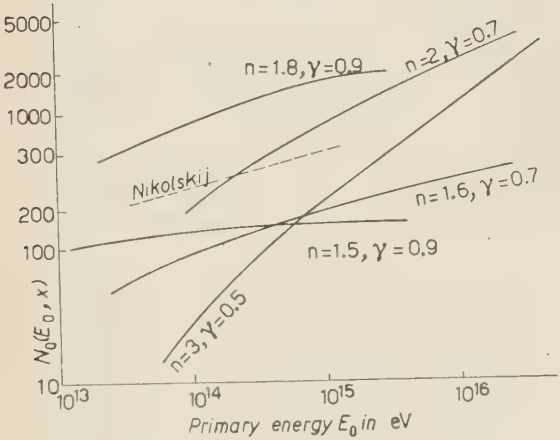

Fig. 1.

The choice of the spectrum (10) for the secondary nucleons in n-$\mathcal{N}$ collisions would appear to be physically unrealistic. But we find that if we choose a different spectrum the results are not much altered provided the values of n and γ are kept approximately the same. This shows that the abundance of the $\mathcal{N}$-component and its variation with energy are very sensitive to n and γ but not so to the spectrum of the secondary nucleons. However, a physically more realistic spectrum would be to assume

$$f_{\mathcal{N}\mathcal{N}}(E, E') = \delta(kE - E') + \bar{n}\,\delta\left(\frac{\gamma - k}{\bar{n}} E' - E\right). \tag{10a}$$

We plot in Fig. 2 the values of N_0 based on the spectrum (10a), choosing $k = 0.8$, $\gamma - 0.9$ and $\bar{n} = 0.6$ and for comparison show the values of N_0 based on (10) for the choice $\gamma = 0.9$ and $n = 1.5$. The close similarity between the two curves is to be noted. The spectrum (10a) implies the existence of a follow-through nucleon carrying 80% of its initial energy and a secondary nucleon once in every two collisions, on the average, carrying about 10% of the energy of the incident particle. This secondary nucleon need not necessarily be a recoil nucleon but could be one of a nucleon-

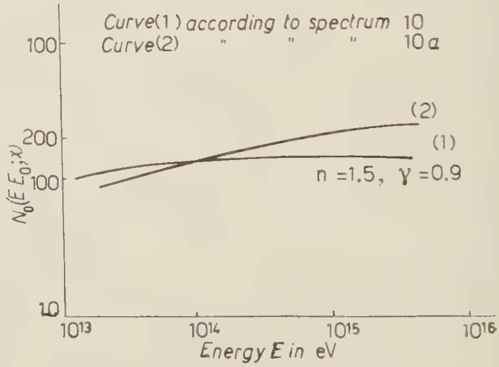

Fig. 2.

pair with a small transverse momentum which gives it a displacement of not more than a few meters from the axis of the shower at the level of observation. Although the spectrum (10a) is physically more realistic we have made our calculations on the basis of (10) since this results in great simplicity in calculations.

To get the contribution due to $\bar{N}(E, E_0, \boldsymbol{x})$ which is equal to $\Pi + \sum_{l=1}^{\infty} N_l$ according to the notation of equations (8) and (9) we make the following choice of the functions involved in these terms:

$$f_{\pi\mathcal{N}}(E, E') = \frac{2}{3}\, m\, \delta\left(\frac{1-\gamma}{m}\, E' - E\right), \tag{15}$$

$$f_{\pi\pi}(E, E') = \delta(k_1 E' - E) + \frac{2}{3}\, m_1\, \delta\left(\frac{\gamma_1 - k_1}{m_1}\, E' - E\right), \tag{16}$$

$$f_{\mathcal{N}\pi}(E, E') = n_1\, \delta\left(\frac{1-\gamma_1}{n_1}\, E' - E\right). \tag{17}$$

For the values of the parameters we take $m = 3$, $m_1 = 6$, $n_1 = 1$, $k_1 = .948$ and $\gamma_1 = .998$. The motivation for this particular choice of the functions used is simplicitly in calculations; the values of the parameters are chosen so as to make the contribution of $\bar{N}$ small compared to N_0 for energies $< 10^{15}$ eV. To achieve this we had to choose rather small multiplicities for the number of pions produced in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions. These pions would represent only the most energetic ones that are emitted in the narrow cone of a « jet » observed in photographic plates; the actual multiplicities could be at least twice as large as the numbers that we have taken. We also find it necessary to include amongst the secondaries produced in a π-$\mathcal{N}$ collision a follow-through pion which carries with it a large part of the energy of the primary particle. We have only one nucleon amongst the secondaries produced in a π-$\mathcal{N}$ collision and it carries only a very small part of the energy of the incident pion. This particularly small value was necessitated because the contribution due to the term $\sum_{l=1}^{\infty} N_l$ is extremely sensitive to the energy imparted to the secondary nucleon produced in a π-$\mathcal{N}$ collision and increases very rapidly with energy. This is connected with the fact that, in the choice of parameters for the n-$\mathcal{N}$ collision made in this paper, two nucleons which may differ in energy by a factor of as much as 100 will produce comparable numbers of secondary nucleons after traversing a certain depth.

Numerical computations. – If we substitute $N_0(E, E', y)$ as given by (11) in (8), all the terms in (8) can be expressed as contour integrals in the complex plane and they can be evaluated by the saddle point method. A similar pro-

cedure is adopted for evaluating the terms in the first series on the R.H.S. of equation (9). The terms in the second series on the R.H.S. of equation (9) represent pions which are produced in π-$\mathcal{N}$ collisions and reach the observer as pions. These terms are evaluated by direct summation. The factor $\pi_l(E', E_0)$ occurring in equations (8) and (9) represents pions of the l-th generation. Because of the particular choice of the parameters in these calculations, only those pions of the l-th generation which, when traced to the primary, involve one or more «consecutive» pion links will make an important contribution to the results. Consequently we evaluated the terms in (8) and (9) successively according to the number of consecutive pion links involved rather than according to the successive generations. The details of this procedure are explained in the Appendix.

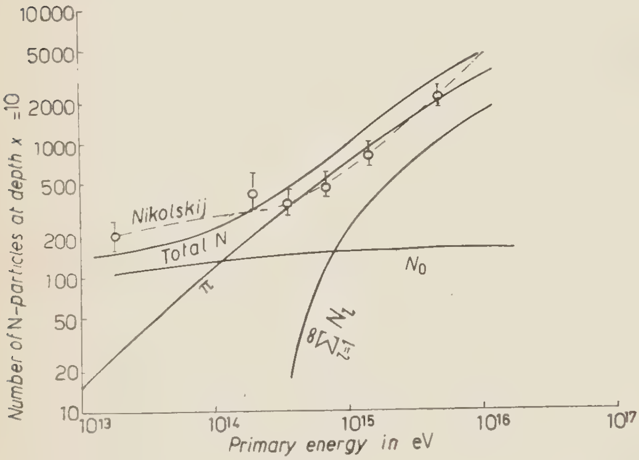

Fig. 3.

4. – Results and discussion.

Our calculations have been carried out for two values of x, namely $x = 10$ (mountain altitude) and $x = 14.7$ (sea level). The results for $x = 10$ and $x = 14.7$ are plotted in Figs. 3 and 4 respectively. The contributions N_0, $\sum_{l=1}^{\infty} N_l$ and Π are separately exhibited and their sum which is the total N-component is plotted. In Fig. 3, the Nikolskij curve is reproduced for comparison. The numbers *calculated* refer to particles having an energy $\geqslant 2$ GeV.

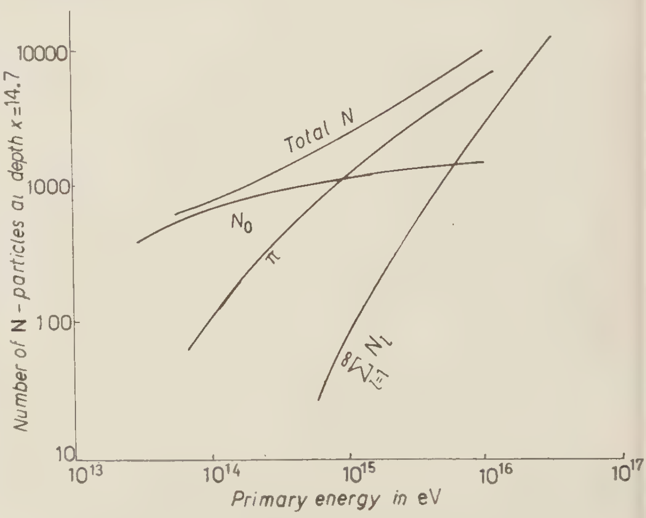

Fig. 4.

The theoretical curve in Fig. 3 roughly reproduces the experimental

curve for the number of $\mathcal{N}$-particles as a function of shower energy: for small showers the increase of the $\mathcal{N}$-component is slow and is essentially due to the term $N_0(E, E_0, \boldsymbol{x})$; pions and those nucleons that are secondary to pions make only a small contribution, since most of the pions decay before they have a chance to interact. At much higher energies, however, the pions have high enough energies to stay in the shower and hence the total $\mathcal{N}$-component rapidly increases. While the general trend of the experimental and theoretical curves is similar, there is no exact fit between them. The difficulty here is that the pions and those nucleons that are secondary to the pions make a relatively small contribution in showers of energy $< 10^{15}$ eV and, at the same time, they make a very large contribution in showers of energy much greater than 10^{15} eV. It was not quite possible to do this so as to get an exact fit between the experimental and theoretical curves by assuming equipartition of energy amongst the secondaries as was done in this paper. The difficulty can be removed if we assume that the energy is distributed amongst the secondaries (both pions and nucleons) in such a way that most of the pions have small energies and the multiplicities of pions produced in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions are somewhat larger; consequently, in showers of moderate energy most of the pions decay and are lost to the nucleon cascade. It is of interest to note in this connection, that TEUCHER *et al.* [6] have found that in « jets » observed in photographic emulsions and which have been ascribed to purely nucleon-nucleon collisions, a surprisingly large number of the pions emitted in a given « jet » carry only a very small part of the energy available in C.G. system.

In going from mountain level to sea level, we find (Fig. 4) that the contribution to the $\mathcal{N}$-component due to the term N_0 is greatly increased while the contribution from other terms is not greatly altered. This is because the nucleons at mountain level represented by the term N_0 are still very energetic and multiply further from mountain level to sea level while the particles represented by the other terms have smaller energies. As a result of this, the rate of increase of the $\mathcal{N}$-component at sea-level should not show such a marked change in going from small showers to big showers as at mountain level. If the sudden increase in the abundance of the $\mathcal{N}$-component observed at mountain-altitude at a shower size N_e (number of charged particles) $\sim 2\cdot 10^5$ (corresponding to a primary energy $\sim 10^{15}$ eV), is attributed to a change in the nature of nuclear interactions or to the role of heavy primaries, we should expect a similar change at sea-level at a shower size corresponding to the *same primary energy*. Assuming that electrons in a shower are absorbed with an attenuation ~ 200 g cm^{-2}, this shower size at sea-level would be $N_e \sim 4\cdot 10^4$. In showers of size $\gg 4\cdot 10^4$ at sea level, the number of $\mathcal{N}$-particles should

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(6) M. W. TEUCHER, D. M. HASKIN and M. SCHEIN: *Phys. Rev.*, **111**, 1384 (1958).

then vary as N_e itself. Recently LEHANE *et al.* [7] have reported observations on the $\mathcal{N}$-component at sea-level in showers whose sizes range from $N_e = 3\cdot10^4$ to $6\cdot10^6$. They state that the number of $\mathcal{N}$-particles varies as $N_e^{0.6\pm0.1}$ over the range of shower sizes investigated by them, and not as N_e, a result which goes against the hypothesis of a change in the nature of nuclear interactions or a cut-off in the primary spectrum. The observations of LEHANE *et al.* imply that over the range of showers investigated there is no *marked* change in the variation of the $\mathcal{N}$-component at sea-level with the size of the shower and that its rate of increase in small showers is somewhat faster than that for small showers at mountain-level. These are, however, just the features that are predicted in this paper on the basis of cascade development of the shower. According to equation (14), we have

$$\frac{\partial \log \mathcal{N}_0(E, E_0; \boldsymbol{x})}{\partial \log E_0} \propto \boldsymbol{x},$$

and hence the rate of increase of the $\mathcal{N}$-component in small showers at sea-level should be faster than that for small showers observed at mountain-level. Moreover, as has already been stated, the $\mathcal{N}$-component in small showers (due to the term $\mathcal{N}_0$) increases considerably from mountain-level to sea-level while in big showers this is not so. The combined effect of these two features is to give a rate of increase of the $\mathcal{N}$-component at sea-level which is faster than that observed in small showers at mountain level without showing a *marked* change in the abundance from small showers to big showers. However, according to the curve in Fig. 3, the rate of increase of the $\mathcal{N}$-component at sea-level in showers of size $N_e > 5\cdot10^6$, would be somewhat faster than that in showers of size $N_e \ll 5\cdot10^6$, *i.e.* on a log-log plot of N *vs.* N_e for sea-level data we would expect a curve which shows a gradual increase in slope as one proceeds to showers of higher size.

An important conclusion that emerges from these calculations is that most of the $\mathcal{N}$-particles in big showers are pions or nucleons that are secondary to pions and hence have comparatively small energies while in small showers most of the $\mathcal{N}$-particles are nucleons which are not secondary to pions and are comparatively more energetic. If one assumes that the electronic component at any given depth is in equilibrium with the $\mathcal{N}$-component, then the electronic component in big showers, since it is mostly derived from the collisions of low energy pions or nucleons that are secondary to pions, will be less penetrating than the electronic component in small showers which is derived from the collisions of energetic $\mathcal{N}$-particles which are not secondary to pions. This appears to be corroborated by the observations of NIKOLSKIJ and Po-

(7) J. A. LEHANE, D. D. MILLAR and M. H. RATHGEBER: *Nature*, **182**, 1699 (1958).

MANSKIJ ([8]) on the absorption of shower particles (mostly electrons) in an absorber consisting of aluminium and graphite. They find that the absorption coefficient of shower particles in big showers is larger than that of shower particles in small showers. For showers with a total number of particles $N_e < 10^5$, the ratio of the number of particles under the absorber N_f to the number of particles above the absorber N_i is equal to $0.55 \pm .035$; for showers with $N_e > 10^5$, this radio is 0.33 ± 0.032.

These observations cannot be explained on the hypothesis of a change in the nature of the elementary act of nuclear interaction, as can be readily seen from the following reasoning:

Let us suppose that nuclear interactions in the energy interval $5 \cdot 10^{12}$ eV to $5 \cdot 10^{14}$ eV are largely elastic while at energies $> 10^{15}$ eV they are largely inelastic and at intermediate values of energy there is a gradual transition from elastic to inelastic collisions. Consider now a shower initiated by a nucleon of energy $\sim 10^{16}$ eV. This nucleon, on colliding, could produce, owing to the highly inelastic character of the collision at these energies, a small number of $\mathcal{N}$-particles of comparable energies. These $\mathcal{N}$-particles again undergo a similar multiplication until the energies of the secondary $\mathcal{N}$-particles have dropped to values of $\sim 5 \cdot 10^{14}$ eV. Once energies of this order are attained the collisions become largely elastic and the subsequent development of the shower is one that could be obtained by the superposition of showers of energies $\sim 5 \cdot 10^{14}$ eV. Hence the penetrating properties of shower particles in a shower of energy $\sim 10^{16}$ eV would be very similar to that of a shower of energy $\sim 5 \cdot 10^{14}$ eV. This would be the case unless a very energetic nucleon, on colliding, splits into some hundreds of $\mathcal{N}$-particles of comparable energy, such that the energy of the primary is distributed, in a few catastrophic processes, amongst a large number of secondary $\mathcal{N}$-particles with comparatively small energies $\sim 10^{12}$ eV. This would avoid the creation of a number of comparatively energetic particles with energies $\sim 5 \cdot 10^{14}$ eV which could degrade their energies only slowly in highly elastic collisions. However, such catastrophic processes involving enormous multiplicities would be highly improbable. In a similar way it can be seen that if the big showers, (numbers of shower particles $> 10^5$), are due to heavy primaries, the absorption of shower particles in such showers cannot be different from that of small showers, (number of shower particles $< 10^5$), if the absorption coefficient in small showers is nearly a constant independent of the size of the shower, as appears to be the case for the shower sizes investigated which differ by a factor of ten.

A result of the present calculations, which can be checked by experiments, is that in small showers the ratio of charged to neutral $\mathcal{N}$-particles is expected to be about unity while in big showers, where the pions are preponderant in

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([8]) S. I. NIKOLSKIJ and A. A. POMANSKIJ: *Žurn. Ėksp. Teor. Fiz.*, **35**, 618 (1958).

the $\mathcal{N}$-component, this ratio would be larger than one. On the hypothesis of a change in the nature of the elementary act of nuclear interaction or that of a high energy cut-off in the energy spectrum of particles (and the consequent preponderance of heavy primary nuclei at very high energies), the ratio of charged to neutral $\mathcal{N}$-particles would be the same for small as well as large showers since, according to these models, big showers may be regarded as a superposition of small showers produced by a unmber of $\mathcal{N}$-particles of comparable energy.

A definite result which follows from these calculations is that the total number of $\mathcal{N}$-particles at sea-level should be larger than the number at mountain-level. This conclusion is fundamental and cannot be avoided; it will always follow as the direct consequence of *any* attempt to explain the observed slow increase of $\mathcal{N}$-particles in small showers. This result, however, seems to contradict our present experimental understanding on the altitude variation of the $\mathcal{N}$-component in air showers. The experimental observations on this point are based on the counting rates at sea-level and mountain altitude of $\mathcal{N}$-particle detector arrays in coincidence with air shower arrays. From the observed increase of the counting rate in going from sea-level to mountain-level it is inferred that $\mathcal{N}$-particles in a shower are absorbed with a long attenuation length exceeding $200\ \mathrm{g\ cm^{-2}}$. However, GREISEN (9) has remarked that an accurate interpretation of the altitude variation of the $\mathcal{N}$-component in air showers must take into account the change of geometry of the showers with elevation. With increasing atmospheric depth the electronic cascades are expected to grow more compressed, while the $\mathcal{N}$-particle cascades would spread outwards. As a result of the greater dispersal of $\mathcal{N}$-particles at sea-level and a reduced counting rate of the shower detector owing to the shrinkage of the electron shower, the counting rate of a system consisting of an $\mathcal{N}$-detector in coincidence with a shower array could be smaller at sea-level than at mountain-level.

A further difficulty in obtaining definite information on the magnitude of the $\mathcal{N}$-component in showers at mountain-level and sea-level is an instrumental one and is related to the efficiency of $\mathcal{N}$-particle detectors for detecting $\mathcal{N}$-particles of different energies. Whilst an $\mathcal{N}$-particle detector may be rated as detecting $\mathcal{N}$-particles of energy > 2 GeV, it could actually have its maximum efficiency only in detecting particles with considerably larger energies, say > 6 GeV. Then, if the multiplication of $\mathcal{N}$-particles from mountain-level to sea-level is mostly in low energy nucleons, the detector would not be able to show up such a multiplication. In concluding on this point, we wish to emphasize that a very slow increase in the number of $\mathcal{N}$-particles with the

(9) K. GREISEN: *Extensive Air Showers - Progress in Cosmic Ray Physics*, vol. **3** (Amsterdam, 1956).

energy of a shower and a strong attenuation of the total number of $\mathcal{N}$-particles in going from mountain-level to sea-level are incompatible; both cannot be valid at the same time.

We have as yet not made any calculations on the electron component in air showers on the basis of our model. We can, however, infer certain general features of the electron component which would result from the present model. From the results of UEDA and OGITA (4) we expect that, on our model which predicts a small inelasticity in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions, the electron component would have a large attenuation length with a strong energy dependence. This result would be in disagreement with the experimental observation that the electrons in air showers are absorbed in the lower atmosphere with a constant attenuation length of nearly 200 g cm^{-2}. Some authors (10,11) have, however, pointed out that fluctuations in the level of the first collision would cause the averaged absorption coefficient to be nearly constant and independent of the size of the shower as well as of the model of shower development. This conclusion has been arrived at only for models which are based on a large inelasticity in n-$\mathcal{N}$ collisions. It remains to be seen whether a similar result holds good for small values of inelasticity in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions.

It is now being recognized that fluctuations have to be taken into account in interpreting air shower phenomena. Fluctuations can be of three types: i) in the charge of the primary particle which initiated the shower; ii) in the level of the first collisions; and iii) in the characteristics of the individual nuclear collisions. We shall consider very qualitatively the general trends which may be brought about by fluctuations of the first two types in the variation of the $\mathcal{N}$-component with the size of the shower. Let us consider first any variation that may result from fluctuations in the charge of the shower-initiating primaries. If we suppose that the $\mathcal{N}$-component in proton-induced showers increases as fast as the energy of the primary particle, then it can be easily seen that the presence of heavy primaries will have no effect on the total number of $\mathcal{N}$-particles in a shower or the variation of the number with the energy of the shower. On the other hand, if the $\mathcal{N}$-component in proton-induced showers increases slowly with the size of the shower, then in heavy primary-induced showers the $\mathcal{N}$-component would increase at a faster rate. The problem is more involved when one considers the effect of fluctuations in the starting points of air showers. Broadly speaking, due to such fluctuations, the average number of particles in a shower due to a primary of given energy could get altered but the rate of increase of $\mathcal{N}$-particles with the size of a shower would remain unaffected. In view of these features, the experimental data on the slow increase of $\mathcal{N}$-particles in small showers should be taken

(10) N. L. GRIGOROV and V. IA. SHESTOPEROV: *Žurn. Èksp. Teor. Fiz.*, **34**, 1539 (1958).
(11) S. MIYAKE: *Progr. Theor. Phys.*, **20**, 844 (1958).

to represent genuine phenomena not caused by fluctuations of the first two types.

One problem which has not been referred to so far and which is of some importance in the context of this paper is the phenomenon of charge exchange in a π-$\mathcal{N}$ collision. According to spectrum (16), we have assumed that amongst the secondary particles produced in the collision of a charged pion there is one charged pion which carries away a large part of the energy of the incident pion. However, we have to consider the possibility that the most energetic pion amongst the secondaries could as well be a neutral pion. According to the present model, charged pions do not play an important role in small showers but they are present in fairly large numbers in big showers. In view of this, the effect of « charge exchange » in a π-$\mathcal{N}$ collision may be adequately accounted for by introducing a factor $\frac{2}{3}$ before the first δ-term on the R.H.S. of equation (16). It would appear that the introduction of such a factor will not greatly alter the results of this paper though it may be necessary to make slight changes in other parameters to maintain the essential consequences.

Since all of the calculations described herein are based on the Nikoslkij curve, we would like to make a few remarks concerning the validity of these experimental observations. The results involve a process of averaging over a large number of showers of a given size in order to obtain the density of $\mathcal{N}$-particles at various distances from the shower axis. The authors claim that the average number of $\mathcal{N}$-particles in a shower obtained thus cannot be in error by more than 10%. In view of the small statistical errors indicated in these observations and the fact that various fluctuations discussed earlier cannot by themselves produce a slow rate of increase of $\mathcal{N}$-particles with the size of the shower, we can be confident that the general trend of the Nikolskij curve is essentially correct and represents a genuine phenomenon. Consequently, the conclusions drawn in this paper can be regarded as significant even though one may not place great significance on an exact fit of the absolute numbers of particles calculated in this paper with the absolute numbers given by Nikolskij *et al.* Further work on the basis of this paper is being carried out to determine the numbers of μ-mesons and electrons in showers of different energies both at mountain and sea-levels.

In concluding, we wish to enumerate the experiments mentioned earlier which are of importance in discriminating between the various possible hypotheses offered to explain the observed features of the $\mathcal{N}$-component in air showers.

1) A measurement of the ratio of neutral to charged $\mathcal{N}$-particles in small showers and in big showers.

2) Measurements of the total number of $\mathcal{N}$-particles as a function of shower size at three or four different altitudes. (On the basis of a change in

the nature of nuclear interactions or a cut-off in the primary spectrum, there should be a sudden change at all these altitudes in the abundance of the $\mathcal{N}$-component at a shower size which should always correspond to the same primary energy ($\sim 10^{15}$ eV). Owing to the attenuation of electrons in a shower, the shower size where this happens decreases with increasing depth in the atmosphere. At shower sizes greater than this value, the number of- $\mathcal{N}$particles should always be proportional to the size of the shower.)

3) An accurate determination of the absorption of shower particles in solid absorbers like graphite at various altitudes. (This is the experiment performed by NIKOLSKIJ and POMANSKY but there is a need to obtain results of greater statistical weight to discriminate between the various explanations.)

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We wish to express our thanks to Professor M. G. K. MENON with whom we had useful discussions on this subject and to Professor B. PETERS who read the manuscript and gave us his comments. We are grateful to Mr. P. K. DAYANIDHI for his assistance in doing numerical calculations.

APPENDIX

Eq. (8) expresses the differential spectrum of nucleons at depth x as a series $\sum_{l=0}^{\infty} N_l(E, E_0; x)$, where

$$N_0(E, E_0; x) = \frac{1}{2\pi i}\frac{1}{E_0}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{E_0}{E}\right)^{s+1} \exp[-\nu(s)x] . \tag{18}$$

$$N_l(E, E_0; x) = \tag{19}$$

$$= \int_0^x dy \exp[-y]\frac{y^l}{l!}\int_E^{E_0} dE'\, N_0(E, E'; x-y)\int_{E'}^{E_0} dE'' f_{\mathcal{N}\pi}(E', E'')\pi_l(E'', E_0) .$$

In eq. (19) the integration with respect to E' can be carried out from 0 to E_0 since $N_0(E, E'; y) = 0$ if $E' < E$, as can be been from eq. (18) by closing the contour by an infinite semicircle to the right. The evaluation of the integral (19) is possible if the production spectra of secondary particles in n-$\mathcal{N}$ and π-$\mathcal{N}$ collisions are represented by homogeneous functions in the primary and secondary energies. Then we can write

$$f_{\mathcal{N}\pi}(E', E'') = \frac{1}{E''} f_{\mathcal{N}\pi}(E'/E'') . \tag{20}$$

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Substituting from (18) and (20) into (19) we get

(21) $N_l(E, E_0; \boldsymbol{x}) =$

$$= \frac{1}{2\pi i}\frac{1}{E_0}\int\limits_{c-i\infty}^{c+i\infty} \mathrm{d}s \left(\frac{E_0}{E}\right)^{s+1} \exp\left[-\nu(s)\boldsymbol{x}\right] \frac{\Gamma[l+1, (1-\nu(s)\boldsymbol{x})]\, R(s)\, V_l(s, E_0)}{l![1-\nu(s)]^{l+1}},$$

where

(22) $$R(s) = \int\limits_0^1 \mathrm{d}u\, u^s f_{\mathcal{N}\pi}(u)\,,$$

(23) $$\left\{\begin{aligned} V_l(S, E_0) &= E_0 \int\limits_0^1 \mathrm{d}v\, v^S \pi_l(vE_0, E_0)\,, \\ \Gamma(l+1, \boldsymbol{x}) &= \int\limits_0^x \exp[-t]\, t^l\, \mathrm{d}t\,. \end{aligned}\right.$$

Thus we have written each term in the serie in (8) as an integral in the complex plane and each of these integrals can be evaluated by the well-known saddle point method. Unless the functions for the production spectra are of a simple type, the functions $V_l(S, E_0)$ given by (23) become complicated functions of S and the evaluation of the integrals by the saddle point method becomes difficult. However, for the δ-type of functions for the production spectra employed in this paper, the function $V_l(S, \boldsymbol{x})$ has a very simple form and the integrals can be easily evaluated.

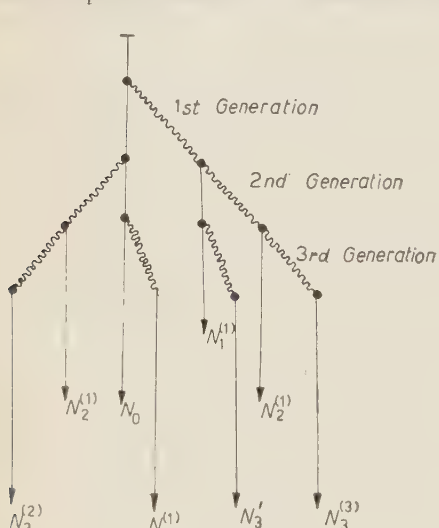

Fig. 5.

The various terms in the series in (8) are represented diagramatically in Fig. 5. The term N_0 is represented as a thick line marked N_0. The term N_1 is represented as another thick line marked $N_1^{(1)}$ which is connected to N_0 by a wavy line which denotes a pion link. The significance of this representation is that if a nucleon in N_1 is taken and traced back to the primary nucleon it will involve one pion link. The term N_2 will be a sum of two terms since π_2 is a sum of two terms. These two terms are denoted by two thick lines marked $N_2^{(1)}$ and $N_2^{(2)}$: the former is connected to N_0 by one pion link and the latter by two consecutive pion links. Similarly N_3 is represented by four lines $N_3^{(1)}$, $N_3^{(2)}$, $N_3^{(3)}$ and N_3'. The line N_3' is connected to N_0 by two pion links which are, however, not consecutive. For the particular choice of the para-

meters which we have made, we find that the contribution from terms of the type N_3' which involve non-consecutive pion links is negligible. The significance of a superscript on N is now obvious. For example $N_l^{(m)}$ denotes a group of nucleons each of which, when traced to the primary nucleon, will involve m consecutive pion links. The subscript implies that these nucleons have developed out of secondary nucleons that have been produced by the collisions of the l-th generation pions. The terms in (8) (wherein we ignore terms of the type N_3') can now be grouped as follows:

$$(24)\qquad \left\{\begin{array}{ll} & N \sim N_0 + N^{(1)} + N^{(2)} + N^{(3)} + \dots \\ \text{where} & \\ & N^{(1)} = N_1^{(1)} + N_2^{(1)} + N_3^{(1)} + N_4^{(1)} + \dots \\ & N^{(2)} \qquad\qquad\quad N_2^{(2)} + N_3^{(2)} + N_4^{(2)} + \dots \\ & N^{(3)} \qquad\qquad\qquad\qquad\quad N_3^{(3)} + N_4^{(3)} + \dots \\ & \text{etc.} \end{array}\right.$$

If terms of the type N_3' also make an important contribution, the above scheme (24) has to be suitably amplified. The scheme of summing according to (24) results in simplification in computational work.

Similar to the superscript on N we can use a superscript on π_l such that $\pi_l^{(m)}$ means that they are l-th generation pions each of which is linked back by $(m-1)$ consecutive pion links. We can easily verify the following recurrence relation with respect to the superscript:

$$(25)\qquad \pi_{l+1}^{(m+1)}(E, E_0) = \frac{1}{1 + B/(l+1)E} \int_E^{E_0} \mathrm{d}E' f_{\pi\pi}(E, E') \pi_l^{(m)}(E', E_0)\,.$$

For the spectrum given by (10), it can be verified that

$$(26)\qquad \pi_{l+1}^{(1)}(E,E_0) = \frac{n^l}{1 + B/(l+1)E} f_{\pi\mathcal{N}}\left(E, \frac{\gamma^l}{n^l} E_0\right).$$

Using (25) and (26) we can determine the functions $V^{(m)}(S, E_0)$, which occur in the integrals for $N_l^{(m)}$. Each of the terms $N_1^{(m)}$, $N_2^{(m)}$, $N_3^{(m)}$, etc. is evaluated for various values E_0 of the primary energy and their sum $N^{(m)}$ at a given energy E_0 is determined graphically. Finally the summation of the $N^{(m)}$'s to give N is also done graphically.

For evaluating the number of pions observed at $\boldsymbol{x}$, we shall write eq. (9) for the differential spectrum of pions as a sum of two series:

$$(27)\qquad \Pi(E, E_0; \boldsymbol{x}) = \sum_{l=1}^{\infty} \Pi_{1,l} + \sum_{l=1}^{\infty} \Pi_{2,l}\,,$$

where

$$(28)\qquad \Pi_{1,l} = \exp[-\boldsymbol{x}]\,x^{-B/E}\int_0^{\boldsymbol{x}} y^{B/E}\exp[y]\,\mathrm{d}y\int_E^{E_0} f_{\pi\mathcal{N}}(E,E')\,N_l(E',E_0;y)\,\mathrm{d}E'.$$

$$(29)\qquad \Pi_{2,l} = \frac{\exp[-\boldsymbol{x}]\,\boldsymbol{x}^l}{l!\,(1+B/lE)}\int_E^{E_0} f_{\pi\pi}(E,E')\,\pi_{l-1}(E',E_0)\,\mathrm{d}E'.$$

In eq. (28), the integration with respect to E' can be carried from E to ∞. Substituting for N_l from eq. (21) in eq. (28) we get

$$(30)\qquad \Pi_{1,l} = \frac{1}{2\pi i E_0}\int_{c-i\infty}^{c+i\infty}\mathrm{d}s\left(\frac{E_0}{E}\right)^{s+1} R(s)\,T(s)\,V_l(s,\boldsymbol{x})\,h_l(s,E;\boldsymbol{x}),$$

where

$$(31)\qquad T(s) = \int_0^1 \mathrm{d}v\, v^s f_{\pi\mathcal{N}}(v),$$

and

$$(32)\qquad h_l(s;E,\boldsymbol{x}) = \boldsymbol{x}^{-B/E}\exp[-\boldsymbol{x}]\int_0^{\boldsymbol{x}} y^{B/E}\exp[(1-\nu(s))\,y]\,\Gamma[l+1,(1-\nu(s))\,y] =$$

$$= \exp[-\nu(s)\boldsymbol{x}]\int_0^{\boldsymbol{x}}\left(1-\frac{v}{\boldsymbol{x}}\right)^{B/E}\exp[-(1-\nu(s))\,v]\,\Gamma[l+1,(1-\nu(s))(\boldsymbol{x}-v)]\,.$$

Since the integrand in (32) vanishes at $v=\boldsymbol{x}$, a very good approximation can be obtained by putting $1-v/\boldsymbol{x}\sim\exp[-v/\boldsymbol{x}]$. Then the integral (32) becomes

$$(33)\qquad h_l(s;E,\boldsymbol{x}) \simeq \exp[-\nu(s)\boldsymbol{x}]\,\frac{\Gamma[l+1,(1-\nu(s))\,\boldsymbol{x}]}{(1-\nu(s))\,[1+B/E[(1-\nu(s))\,\boldsymbol{x}]]} -$$

$$-\exp[-\boldsymbol{x}]\,(1-\nu(s))^l x^l\int_0^{(1-\nu(s)\boldsymbol{x})}\exp\left[-\frac{By}{E(1-\nu(s))\,\boldsymbol{x}}\right]\left[1-\frac{y}{(1-\nu(s))\boldsymbol{x}}\right]^l\mathrm{d}y \simeq$$

$$\simeq \frac{\exp[-\nu(s)\boldsymbol{x}]\,\Gamma[l+1,(1-\nu(s))\,\boldsymbol{x}]}{(1-\nu(s))\,[1+B/[E(1-\nu(s))\boldsymbol{x}]]} - \frac{\exp[-\boldsymbol{x}](1-\nu(s)\boldsymbol{x})^{l+1}}{(1-\nu(s))\,[1+B/[E(1-\nu(s))\boldsymbol{x}]]\,(l+B/E)}\,.$$

Except for large values of l, the second term in (33) will be small compared to the first term. Taking only the first term in (33) we can write eq. (30) as

$$(34)\qquad \Pi_{1,l}(E,E_0;\boldsymbol{x}) =$$

$$= \frac{1}{2\pi i E_0}\int_{c-i\infty}^{c+i\infty}\mathrm{d}s\left(\frac{E_0}{E}\right)^{s+1}\frac{R(s)\,T(s)\,V_l(s,E_0)\exp[-\nu(s)\boldsymbol{x}]\,\Gamma[l+1,(1-\nu(s))\,\boldsymbol{x}]}{l!\,(1-\nu(s))^{l+2}[1+B/[E(1-\nu(s))\,\boldsymbol{x}]]}\,.$$

Eq. (34) is very similar to eq. (21) except that in (34), E occurs in a more complicated way than in (21). We have to remember that we have still to

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make an integration with respect to E in (21) and (34) to get the integral number of nucleons.

As we had already mentioned, the second series in (27) is summed up directly.

We may mention that the method of obtaining the number of nucleons and pions according to the solutions given by (8) and (9) involves, whenever this is possible, a more rapid convergence of the series than the usual one by the method of successive collisions as given by eqs. (4) and (5). We believe that the method developed in this paper will also be useful otherwise as it gives some physical insight into the development of the cascade of nucleons and mesons in terms of the pion links.

RIASSUNTO (*)

Secondo NIKOLSKIJ *et al.* (1957), il numero di particelle dotate di attività nucleare ($\mathcal{N}$ negli sciami in aria cresce abbastanza lentamente, in proporzione ad $E^{0.2}$, per gli sciam aventi energie comprese fra 10^{13} eV e 10^{15} eV. Ad energie $> 10^{15}$ eV l'accrescimento è più rapido ed è $\propto E$. Questa variazione della componente $\mathcal{N}$ con le dimensioni dello sciame ed altre caratteristiche riscontrate negli sciami in aria sono state precedentemente interpretate come indicazioni di un cambiamento nella natura delle interazioni nucleari ad energie molto elevate o come indicazioni di particolarità speciali nella natura e composizione della radiazione cosmica primaria, quali l'esistenza di cut-off nella rigidità magnetica delle particelle ed il predominio di primari pesanti ad energie molto elevate. In questo scritto si tenta di spiegare le suddette particolarità osservate con alcune caratteristiche delle collisioni fra nucleone e nucleo (n-$\mathcal{N}$) e fra pione e nucleo (π-$\mathcal{N}$); il termine nucleo si riferisce qui al «nucleo d'aria». L'aspetto principale della curva (di Nikolskij) relativa all'abbondanza di particelle $\mathcal{N}$ rispetto all'energia dello sciame, che qui si prende in considerazione, non è il rapido accrescimento della componente $\mathcal{N}$ in sciami di energia $> 10^{15}$ eV, nè l'esistenza di una variazione nell'abbondanza di particelle $\mathcal{N}$ passando da piccoli sciami a grandi sciami, ma piuttosto l'aumento relativamente lento del numero delle particelle $\mathcal{N}$ negli sciami di energia di $(10^{13} \div 10^{15})$ eV. Questo lento aumento, che è in contrasto con l'aspettativa generale nel calcolo delle cascate, pone rigide limitazioni alla scelta dei parametri che caratterizzano le collisioni n-$\mathcal{N}$ e π-$\mathcal{N}$. Si mostra che le collisioni n-$\mathcal{N}$ e π-$\mathcal{N}$ sono largamente elastiche e che il numero di nucleoni secondari prodotti in queste collisioni è molto piccolo (dell'ordine di uno o due); per di più, in una collisione π-$\mathcal{N}$, l'energia che passa nella componente nucleonica è una frazione molto piccola dell'energia del pione incidente. Queste particolarità portano pertanto al risultato di un rapido accrescimento della componente $\mathcal{N}$ nei grandi sciami; questo nasce come risultato del ruolo complicato che esplicano i pioni carichi per la loro vita media finita. Nei piccoli sciami i pioni elettricamente carichi hanno solo piccole energie e quindi nella maggior parte decadono e non danno un contributo apprezzabile alla componente $\mathcal{N}$. Nei grandi sciami, invece, i pioni hanno energie sufficientemente elevate per produrre interazioni nucleari prima di decadere e per la grande molteplicità di pioni prodotti nelle collisioni n-$\mathcal{N}$ e π-$\mathcal{N}$, si ha un rapido accrescimento nel numero delle particelle $\mathcal{N}$ all'aumentare dell'energia dello sciame. Si suggeriscono alcuni esperimenti per poter fare distinzione fra le varie spiegazioni possibili.

(*) *Traduzione a cura della Redazione.*

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On the Relative Abundance of Carbon, Nitrogen and Oxygen in the Cosmic Rays.

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(ricevuto il 21 Marzo 1960)

Summary. — Plates (33 G-5 emulsions) were exposed at an atmospheric depth of approximately 6 g/m². Gap length measurements of 220 medium nuclei have these ratios C:N:O::1.5:0.97:1.0. By using different parameters in the diffusion equation we were unable to obtain agreement with the values of Suess and Urey's cosmic abundances. The results indicate that the chemical composition of the sources of cosmic rays is different from the mean composition of the Universe.

1. – Introduction.

During the last years a considerable amount of information has been collected on the charge composition of the cosmic radiation incident on the top of the atmosphere (1-8).

Recent advances in the technique of balloon flying made it possible to expose large emulsion stacks at very high altitudes, in order to reduce the uncertainties in the extrapolation from the observed fluxes to the fluxes at the top of the atmosphere. A satisfactory degree of consistency has been

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(1) V. Y. Rajopadhye and C. J. Waddington: *Phil. Mag.*, **3**, 19 (1958).

(2) A. Engler, M. F. Kaplon and J. Klarman: *Phys. Rev.*, **112**, 597 (1958).

(3) P. S. Freier, E. P. Ney and C. J. Waddington: *Phys. Rev.*, **113**, 921 (1959).

(4) C. M. Garelli, B. Quassiati and M. Vigone: *Nuovo Cimento*, **15**, 121 (1960).

(5) C. J. Waddington: *Phil. Mag.*, **2**, 1059 (1957).

(6) H. Koshiba, G. S. Schultz and M. Schein: *Nuovo Cimento*, **9**, 1 (1958).

(7) P. H. Fowler, R. R. Hiller and C. J. Waddington: *Phil. Mag.*, **2**, 293 (1957).

(8) M. V. K. Appa Rao, S. Biswas, R. R. Daniel, K. A. Neelakantan and B. Peters: *Phys. Rev.*, **110**, 751 (1958).

reached in this way by different experiments in the flux values of the three general groups in which heavy nuclei are usually divided, as we can see from the following Table I.

TABLE I. – *Fluxes in* particles/m² sr s *at the top of the atmosphere.*

Author	Light nuclei (L) $3 \leqslant z \leqslant 5$	Medium nuclei (M) $6 \leqslant z \leqslant 9$	Heavy nuclei (H) $z \geqslant 10$	Year
Bristol (1)	2.3 ± 0.4	6.1 ± 0.6	2.5 ± 0.3	1957
Rochester (2)	0.5 ± 0.4	6.1 ± 0.6	2.2 ± 0.4	1958
	1.7 ± 0.3	5.6 ± 0.6	2.2 ± 0.4	
Minnesota (3)	1.9 ± 0.3	5.1 ± 0.5	1.7 ± 0.3	1959
Turin (4)	1.9 ± 0.3	4.4 ± 0.4	1.6 ± 0.2	1959

It is apparent from the last works the necessity of a more detailed knowledge of the relative abundance of elements in the cosmic radiation at the top of the atmosphere, if we aim to investigate about the origin of the cosmic rays.

The purpose of the present work is to add some results to the relative abundance of medium elements (C, N, O).

2. – Exposure details and charge measurements.

The emulsion stack used in this research is the same used by the Turin group (4); it has been exposed over Texas at a geomagnetic latitude of 41° N in September 1958, at an atmospheric depht of about 6 g/cm².

Using the same criteria described in the above mentioned paper (4) (which we shall refer to as paper III) a scanning has been done in 33 G-5 emulsions 3 cm below the top edge, for tracks produced by heavy nuclei of $Z \geqslant 5$, and a total of 238 medium nuclei was found.

As we were concerned principally in the medium component of the spectrum, the gap length method of charge measurements was used. As described in paper III, measurements were performed, whenever possible in four different sheets (2 G-5, and 2 L-4) of the stack.

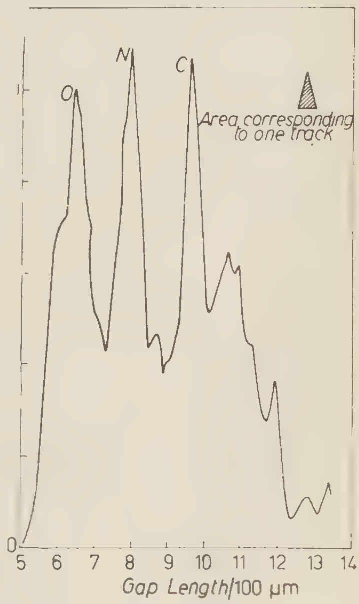

Fig. 1.

As can be seen in the hystogram of Fig. 1, the resolution obtained in G-5 emulsion is very good, so that for a very large part of the tracks the charge could be assigned with an uncertainly $\Delta Z < 1$.

Only for a 5% of the tracks, owing to unfavorable geometrical conditions it was not possible to obtain the resolution between neighbouring tracks; they were not included in the statistics used to evaluate the relative abundances of C, N, O, nuclei, although they where used when calculating the extrapolated flux values of the three charge groups L, M and H.

3. – Experimental results and discussion.

The results of charge determination on the 220 tracks available for gap-lenght measurement giving the resolution $\Delta Z < 1$ are the following:

C	96 tracks
N	61 »
O	63 »

These results are in satisfactory agreement with those quoted by other groups as shown in Table II.

TABLE II.

GROUP	Relative abundance of C, N, O
R (2)	C:N:O = 2.0:1.2:1.0
B (5)	C:N:O = 1.8:1.2:1.0
C (6)	C:N:O = 1.6:1.1:1.0
W (9)	C:N:O = 1.8:1.1:1.0
Present work	C:N:O = 1.5:0.97:1.0
SUESS and UREY (10) Cosmic Abundances	C:N:O = 0.16:0.31:1.0

On the contrary, there is a striking discrepancy between the cosmic rays data and the data derived from the cosmic abundance of elements given by SUESS and UREY (10).

We tried to see if the observed C:N:O ratios at the top of the atmosphere could be put into agreement with the hypothesis that the values of the ratios at the cosmic rays sources are the same of Suess and Urey cosmic abundances (10), by a suitable choice of the parameters entering in the diffusion equa-

(9) C. FICHTEL: private communication.
(10) H. E. SUESS and M. C. UREY: *Rev. Mod. Phys.*, **28**, 53 (1956).

tion. As the knowledge of these data is at present, very poor, we tried three different sets of fragmentation probabilities in interstellar matter:

	set I	set II	set III
p_{LL}	0.07	0.10	0.04
p_{MM}	0.09	0.11	0.07
p_{ML}	0.34	0.38	0.30
p_{HH}	0.29	0.34	0.24
p_{HL}	0.27	0.34	0.20
p_{HM}	0.36	0.40	0.32

Set I is the weighted average of the Chicago (6) and Bristol (1) data, the others sets are maximum and minimum values, chosen to give a reasonable upper and lower limits for the mean path d traversed by the cosmic radiation. We assumed for interstellar matter, a density of 0.1 atoms/cm³ and a composition of 90% H and 10% He.

For the fluxes values at the top of the atmosphere we used the mean of the data of the Minnesota (3) and Turin (4) group, that is:

$$N^0_L = 1.9 \text{ particles/m}^2 \text{ sr s}$$

$$N^0_M = 4.7 \qquad \text{»}$$

$$N^0_H = 1.7 \qquad \text{»}$$

and for the means free paths of L, M and H nuclei in interstellar matter:

$$\lambda_L = 5.78 \cdot 10^{25} \text{ cm}$$

$$\lambda_M = 3.88 \cdot 10^{25} \text{ cm}$$

$$\lambda_H = 2.08 \cdot 10^{25} \text{ cm} .$$

With the hypothesis that at the origin the flux of light nuclei is zero, we obtained in the three cases the following results:

Set of fragmentation probabilities used	Mean path traversed (d) $\cdot 10^{-25}$ cm	Ratio between the heavy and medium elements at the origin (Q_H/Q_M)
III (minimum)	3.3	0.70
I (medium)	2.7	0.58
II (maximum)	2.4	0.56

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Moreover we made the following assumptions:

1) In origin, the relatives abundances of medium elements are in accordance with the universal chemical abundance data of SUESS and UREY [10].

2) In interstellar matter:

a) $\lambda_M = \lambda_N = 3.88 \cdot 10^{25}$ cm = mean free path of N nuclei

$\lambda_O = 3.56 \cdot 10^{25}$ cm = mean free path of O nuclei

$\lambda_C = 4.31 \cdot 10^{25}$ cm = mean free path of C nuclei

b) The total fragmentation probabilities are equally subdivided in the fragmentation probabilities of the singles medium elements (C, N, O, F), *i.e.*,

$$p_{HM} \text{ is subdivided in } p_{HC} = p_{HN} = p_{HO} = p_{HF} = \tfrac{1}{4} p_{HM}$$

$$p_{MM} \text{ is subdivided in } p_{CC} = p_{ON} = p_{NC} = \tfrac{1}{3} p_{MM} \, .$$

c) The fragmentations probabilities

$$p_{FF}\,, \quad p_{FO}\,, \quad p_{FC}\,, \quad p_{OO}\,, \quad p_{NN}\,, \quad p_{CC} \text{ are negligeable.}$$

(Assumptions *b*) and *c*) are reasonably supported by an analysis of all the interactions reported by the Bristol [1] and Turin [11] groups.)

The results on the relative abundances of C, N, O nuclei obtained for the three sets of data, at the top of the atmosphere are reported in Table III. (The latter line refers to results obtained with the data of SUESS and UREY [10], for the relative abundances of the elements, in the source of cosmic rays.)

If we compare these values (Table III) with the experimental ones of Table II, we conclude that the discrepancy cannot be ascribed to the uncer-

TABLE III.

Set of data used (fragm. prob.; d; Q_H/Q_M)	Relative abundance of C, N, O
set III; $3.3 \cdot 10^{25}$ cm; 0.70	C:N:O = 0.27:0.44:1.0
set II; $2.4 \cdot 10^{25}$ cm; 0.56	C:N:O = 0.21:0.47:1.0
set I; $2.7 \cdot 10^{25}$ cm; 0.58	C:N:O = 0.25:0.38:1.0
set I; $2.7 \cdot 10^{25}$ cm; 0.375	C:N:O = 0.15:0.37:1.0

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[11] R. CESTER, A. DEBENEDETTI, C. M. GARELLI, B. QUASSIATI, L. TALLONE and M. VIGONE: *Nuovo Cimento*, 7, 371 (1958).

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tainties in the values of the parameters involved in the methods of extrapolation. Most probably the chemical composition of the sources of cosmic rays is rather different from the mean composition of Universe.

* * *

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RIASSUNTO (*)

Delle lastre (33 emulsioni G-5) furono esposte ad una profondità atmosferica di circa 6 g/cm^2. Le misure della lunghezza di gap di 220 nuclei medi hanno dato questi rapporti: C:N:O::1.5:0.97:1.0. Usando differenti parametri nell'equazione di diffusione non abbiamo potuto ottenere concordanza con i valori delle abbondanze cosmiche di Suess ed Urey. I risultati indicano che la composizione chimica delle sorgenti di raggi cosmici è diversa dalla composizione media dell'universo.

(*) *Traduzione a cura della Redazione.*

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Prong Analysis of Multiple Production of Pions at Bevatron Energy According to a Statistical Theory.

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(ricevuto il 24 Marzo 1960)

Summary. — A model is proposed in which « central » collisions are treated according to a statistical theory and « peripheral » collisions according to a simple two-centre model. The relative probability of the two kinds of collisions is introduced as a new parameter. With this one additional parameter it is possible to have agreement with the experimental values of the inelasticity, the average number of pions, the ratio of two- to four- to six-prong events, and the average energies. The introduction of a π-π isobar changes the result, but present experiments, because of lack of precision, cannot confirm or rule out this hypothesis. The model predicts also angular distributions and some angular correlations, as well as energy spectra. In the first part of the present investigation it is shown that a purely statistical model (without « peripheral » collisions) fails to account for the rather low inelasticity which is experimentally found, because it yields too high meson energies and much too low nucleon energies.

1. – Introduction.

In a recent paper (1) R. Hagedorn has applied a statistical theory of particle production to events resulting from p-p collisions at Bevatron energies. The results, as far as the average number of produced pions is concerned, agree well with experiment (2-4). It would, however, be much desirable to have

(1) R. Hagedorn: *Nuovo Cimento*, **15**, 246 (1960).

(2) R. M. Kalbach, J. J. Lord and C. H. Tsao: *Phys. Rev.*, **113**, 330 (1959).

(3) R. R. Daniel, N. Kameswara Rao, P. K. Malkotra and Y. Tsuzuki: preprint from Tata Institute, Bombay.

(4) H. Winzeler: private communication.

further points of comparison, as *e.g.* spectra of particles coming from events with the same multiplicity. In the statistical theory particles with the same mass have the same form of spectrum, for a given type of event. All pions (of charge +, 0, −) produced for example in a collision of the type $p+p \to \to 2\mathcal{N}+3\pi$ will show the same spectrum. There are now many different possibilities to distribute the charges over 2 nucleons and 3 pions:

pp	+ − 0
	0 0 0
pn	+ + −
	+ 0 0
nn	+ + 0

One sees that the same type of event (in the example: production of three mesons) will appear to the observer in many different ways. For instance as events with two protons, one proton, no proton, or as events with 2 prongs, 4 prongs, 6 prongs. Other types of events (*e.g.* 4 meson production) which give different spectra will appear also to the observer as 2, 4 or 6 prong events. As it is almost impossible in most cases to ascertain the number of neutrals in a star, one cannot compare directly the theory and the experiment.

However, because of isospin conservation, it is possible to compute the relative probabilities of different charge-states for events with a given multiplicity. One can therefore compute the relative weight of 2, 4, 6 prong stars for each multiplicity. The statistical theory, in turn, gives probabilities and spectra for every multiplicity. Combining the two informations, it is possible to derive the spectra of protons and charged pions in 2, 4, 6 or 8 prong events, and thus arrive at data which are directly comparable with experiment.

A detailed comparison with the experimental data shows then an important discrepancy: the inelasticity predicted by the theory is 75 %, the experimental figure is 49%. We have tried to explain this by treating the glancing p-p collisions with a so-called « two-centre » model, and using the statistical model only for the central collisions. The ratio of central to glancing collisions is introduced as a parameter.

The success of the π-isobar in the statistical theory of $\bar{p}$-p annihilation has led us to consider a π^* also in the meson production from p-p collisions. The result is that this does not contradict the experimental data; these are however too inaccurate at present to allow definite conclusions to be drawn about the π^* in these phenomena.

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2. – Calculation of charge distribution in the statistical theory.

The formula—which was explained in ([1]) and ([5])—giving the probability of an event of type b is

$$(1)\qquad P_b = W_{\alpha,\beta,\gamma}\,(T)\left\{\frac{\prod(2s_i+1)^{N_i}}{\prod N_i!}\right\}_b \Omega^{n-1}\,\varrho_b^*(E, m_1, \dots, m_n; p=0)\,,$$

where the event b is characterized by stating the number N_i of particles with mass m_i, spin s_i and isospin t_i. The coefficient $W_{\alpha,\beta,\gamma}$ depends on the total isospin and on the t_i; $W_{\alpha,\beta,\gamma}$ is equal to the number of independent isospin eigenfunctions to a total isospin T that can be built from α isospin $\frac{1}{2}$ particles β isospin 1 and γ isospin $\frac{3}{2}$ particles. Tables of $W_{\alpha,\beta,\gamma}$ have been published ([6,7]).

We see that b stands for a whole set of events, which can be distinguished from each other by different charge distributions of the particles. One may ask *e.g.* for the probability that in an event of type $b = 2$ nucleons$+n$ mesons one will find a proton, a neutron and numbers n_+, n_0, n_- of π^+, π^0, π^- respectively. This probability is obtained by replacing in (1) the coefficient $W_{2,n,0}$ by another coefficient

$$^{\frac{1}{2},\frac{1}{2},1,\dots,1}{}^*P^1_{\frac{1}{2},-\frac{1}{2};n_+,n_0,n}\;.$$

The coefficients $^{\dots}P^1_{\dots}$ have been derived and computed for all interesting cases ([8]). The sum of all the $^{\dots *}P^1_{\dots}$ over all possible charges (lower indices) is equal to $W_{2,n,0}$, $(^{\dots *}P^1_{\dots}/W_{\alpha,\beta,\gamma})$ is the relative probability of a given charge state in events of type b.

A slight complication arises in our problem because of the production of nucleon isobars (*) which decay fast and are observed as nucleons and pions, whose spectrum is different from the spectrum of the directly produced nucleons and pions. For instance, the events where 2 nucleons and 3 pions are

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([5]) F. Cerulus and R. Hagedorn: CERN Report 59-3.

([6]) Y. Yeivin and A. de-Shalit: *Nuovo Cimento*, **1**, 1147 (1955).

([7]) V. S. Barašenkov and B. M. Barbašev: *Suppl. Nuovo Cimento*, **7**, 19 (1957).

([8]) F. Cerulus: *Suppl. Nuovo Cimento*, **15**, 402 (1960).

(*) The introduction of a $\mathcal{N}^*$ is discussed in ([9]) and ([5]). It corresponds to the $\frac{3}{2}\,\frac{3}{2}$ resonance in π-$\mathcal{N}$ scattering. Its mass is therefore taken as $1.31\,M_{\mathcal{N}}$ with $T=\frac{3}{2}$, $J=\frac{3}{2}$.

([9]) S. Z. Belen'kij, V. M. Maksimenko, A. I. Nikišov and I. L. Rozental': *Usp. Fiz. Nauk*, **62** (2), 1 (1957), transl. in *Fortschr. d. Phys.*, **6**, 524 (1958).

produced come from three different types b of collisions, *viz.*

$$(2)\qquad \begin{cases} 2\mathcal{N} + 3\pi\,, \\ \mathcal{N} + \mathcal{N}^* + 2\pi \rightarrow \mathcal{N} + (\mathcal{N} + \pi) + 2\pi \\ 2\mathcal{N}^* + \pi \qquad \rightarrow 2(\mathcal{N} + \pi) + \pi\,. \end{cases}$$

Because $\mathcal{N}^*$ has isospin $\frac{3}{2}$, some charge states of systems with one or more $\mathcal{N}^*$ give rise to several charge states of the end-products, according to Table I.

TABLE I.

t_3 of N^*	Decay products	Branching ratio
$\frac{3}{2}$	$\pi^+ + p$	1
$\frac{1}{2}$	$\pi^+ + n$	$\frac{1}{3}$
	$\pi^0 + p$	$\frac{2}{3}$
$-\frac{1}{2}$	$\pi^0 + n$	$\frac{2}{3}$
	$\pi^- + p$	$\frac{1}{3}$
$-\frac{3}{2}$	$\pi^- + n$	1

So, having calculated the probability of charge states for each type b (which involve 0, 1 or 2 $\mathcal{N}^*$), one has to go a step further and compute from them the probabilities for a given observable charge state (*i.e.* $2\mathcal{N}$ and $n\pi$).

This has been described elsewhere ([5]) but for the sake of clarity we shall treat the above example in detail. We shall denote by

$w_{b,\pi}(\varepsilon)$ the spectrum of the pions produced directly in an event b

$w_{b,\pi'}(\varepsilon)$ the spectrum of the pions produced by $\mathcal{N}^*$ in an event b

$w_{b,\mathcal{N}}(\varepsilon)$ the spectrum of the nucleons produced directly in an event b

$w_{b,\mathcal{N}'}(\varepsilon)$ the spectrum of the nucleons produced by $\mathcal{N}^*$ in an event b

We number the events (2) by $b=0$, $b=1$, $b=2$, the $w_{b,\dots}$ are normalized in such a way that

$$P_b = W_{\alpha,\beta,\gamma\cdot}\int\limits_0^{\varepsilon_{\max}} w_b(\varepsilon)\,\mathrm{d}\varepsilon\,.$$

The charge states for the $b = 0,\, 1,\, 2$ are given in Table II.

Suppose one wants to compute the spectra of pions from two prong events with one proton. One selects therefore end states with one proton and one charged pion.

TABLE II.

b	States with isobars: t_3 baryons	States with isobars: t_3 pions	$^{\dots *}P^1_{\dots}$	Observable end states: nucleons direct	nucleons from $\mathcal{N}^*$	π direct	π from $\mathcal{N}^*$	weight of state	Serial number of end states
0	$\frac{1}{2}$ $\frac{1}{2}$	+ 0 −	2.400	p p		+ 0 −		2.400	1
		0 0 0	0.300	p p		0 0 0		0.300	2
	$\frac{1}{2}$ $-\frac{1}{2}$	+ 0 0	2.100	n p		+ 0 0		2.100	3
		+ + −	3.000	n p		+ + −		3.000	4
	$-\frac{1}{2}$ $-\frac{1}{2}$	+ + 0	1.200	n n		+ + 0		1.200	5
1	$\frac{3}{2}$ $\frac{1}{2}$	0 −	0.800	p	p	0 −	+	0.800	6
	$\frac{3}{2}$ $-\frac{1}{2}$	0 0	0.350	n	n	0 0	+	0.350	7
			1.000	n	p	+ −	+	1.000	8
	$\frac{1}{2}$ $\frac{1}{2}$	0 0	0.250	p	p	0 0	0	0.167	9
				p	n	0 0	+	0.083	10
		+ −	0.600	p	p	+ −	0	0.400	11
				p	n	+ −	+	0.200	12
	$\frac{1}{2}$ $-\frac{1}{2}$	+ 0	1.200	n	p	+ 0	0	0.800	13
				n	n	+ 0	+	0.400	14
	$-\frac{1}{2}$ $-\frac{1}{2}$	+ +	0.300	n	p	+ +	−	0.100	15
				n	n	+ +	0	0.200	16
	$-\frac{3}{2}$ $\frac{1}{2}$	+ +	0.500	p	n	+ +	−	0.500	17
2	$\frac{3}{2}$ $\frac{1}{2}$	−	0.600		p p	−	+ 0	0.4	18
					p n	−	+ +	0.2	19
	$\frac{3}{2}$ $-\frac{1}{2}$	0	0.600		p p	0	+ −	0.2	20
					p n	0	+ 0	0.4	21
	$\frac{3}{2}$ $-\frac{3}{2}$	+	1.000		p n	+	+ −	1.000	22
	$\frac{1}{2}$ $\frac{1}{2}$	0	0.200		p p	0	0 0	0.089	23
					p n	0	+ 0	0.089	24
					n n	0	+ +	0.022	25
	$\frac{1}{2}$ $-\frac{1}{2}$	+	0.600		p p	+	− 0	0.133	26
					p n	+	0 0	0.267	27
					p n	+	+ −	0.067	28
					n n	+	+ 0	0.133	29

For

$b = 0$ this is state n^r: 3

$b = 1$ these are states n^r: 7, 10, 13

$b = 2$ » » » n^r: 21, 24, 27

$w_{0,\pi}$ is therefore multiplied with the weight 2.100
$w_{1,\pi}$ » » » » » » 0.800
$w_{2,\pi}$ » » » » » » 0.267
$w_{1,\pi'}$ » » » » » » (0.350 + 0.083)
$w_{2,\pi'}$ » » » » » » (0.400 + 0.089)

and all these weighted spectra are added, together with the other weighted spectra from end states with 1, 2, 3, 5 ... pions. These other weights are obtained in a similar way, by combining the coefficients $^{\cdots *}P^1_{\cdots}$ from (8) with the known branching ratios of the $\mathcal{N}^*$.

In practice the spectra for the events labelled b (*i.e.* phase space calculations) are computed by an electronic computer (10), and the calculation outlined in the previous paragraph is done electronically too (11). The calculation has been organized in such a way that the end states are distinguished first by the number of charged particles (prongs). For a given number of prongs one distinguishes further between stars with 0, 1 or 2 protons, and in each case the spectrum of the $\pi^\pm$ and the p is computed (*). Once these spectra are known, it is easy to obtain any other kind of spectrum, to compare with experiment. *E.g.* the total p-spectrum, or the spectrum of $\pi^\pm$ in 4 prong events, etc.

TABLE III. – *Probabilities of different prong numbers in inelastic* p-p *collisions at* 6.2 GeV (*neglecting* K-*meson production*), *for different values of the interaction volume* Ω; $\Omega_0 = (4\pi/3)(\hbar/\mu c)^3$, $L = 2M/E$. The results are given: a) assuming a pion-nucleon isobar $\mathcal{N}^*$ b) assuming a pion-nucleon isobar $\mathcal{N}^*$ and a pion-pion isobar π^*.

number of prongs	probability with $\mathcal{N}^*$			probability with $\mathcal{N}^*$ and π^*		
	$\Omega=2L\Omega_0$	$\Omega=L\Omega_0$	$\Omega=\frac{1}{2}L\Omega_0$	$\Omega=2L\Omega_0$	$\Omega=L\Omega_0$	$\Omega=\frac{1}{2}L\Omega_0$
2	0.356	0.434	0.502	0.215	0.255	0.303
4	0.575	0.526	0.476	0.643	0.635	0.616
6	0.068	0.040	0.022	0.142	0.110	0.081
8	$3\cdot10^{-4}$	$7\cdot10^{-5}$	$1.5\cdot10^{-5}$	$1\cdot10^{-3}$	$7\cdot10^{-4}$	$4\cdot10^{-4}$

(10) F. CERULUS and R. HAGEDORN: *Suppl. Nuovo Cimento*, **9**, 646, 659 (1958).
(11) R. HAGEDORN: CERN Report 59-25.
(*) For conciseness they are not given here, but they are available upon request.

The multiplicities and average energies are displayed in Tables III and IV. An analysis which can be compared directly with experiment (2) has been carried out, and is displayed in Figs. 1-12 (full line).

TABLE IV. – *Average particle numbers* $\langle n \rangle$ *and average kinetic energies* $\langle \varepsilon_{\text{kin}} \rangle$ (*in units* 0.938 MeV) *of nucleons and pions from* 6.2 GeV p-p *collisions in different types of events for* $\Omega = (2M/E)\Omega_0$.

Type of event			Assuming $\mathcal{N}^*$				Assuming $\mathcal{N}^*$ and π^*			
number of prongs	number of protons	number of $\pi^\pm$	$\langle n \rangle_p$	$\langle \varepsilon_{\text{kin}} \rangle_p$	$\langle n \rangle_{\pi^\pm}$	$\langle \varepsilon_{\text{kin}} \rangle_{\pi^\pm}$	$\langle n \rangle_p$	$\langle \varepsilon_{\text{kin}} \rangle_p$	$\langle n \rangle_{\pi^\pm}$	$\langle \varepsilon_{\text{kin}} \rangle_{\pi^\pm}$
2	2	0	0.117	405		496	0.0315	405		
	1	1	0.289	333	0.289	496	0.160	323	0.160	430
	0	2			0.182	435			0.161	350
4	2	2	0.446	285	0.446	424	0.445	264	0.445	349
	1	3	0.282	225	0.847	347	0.378	204	1.14	289
	0	4			0.107	370			0.137	269
6	2	4	0.0671	173	0.134	284	0.160	164	0.319	249
	1	5	0.00638	133	0.0319	225	0.030	126	0.150	216
	0	6			$2.28\cdot 10^{-4}$	176			$2.05\cdot 10^{-3}$	201
8	2	6	$1.41\cdot 10^{-4}$	103	$4.22\cdot 10^{-4}$	176	$1.34\cdot 10^{-3}$	89	$4.0\cdot 10^{-3}$	185
	1	7	$8.91\cdot 10^{-7}$	84	$6.23\cdot 10^{-6}$	136	$2.4\cdot 10^{-7}$	84	$1.7\cdot 10^{-6}$	136

In all these calculations strange particle production has been neglected. This is justified, we think, because of the rather small probability with which K-meson events appear. According to the statistical theory (1) we neglect 7.2 % of all reactions. It would of course be possible to include the strange particles in the prong analysis. This would, however, involve much more calculations, while it is unnecessary to obtain a comparison with experiment. Indeed it should be possible to reject from the statistics most of the stars leading to a V-particle.

3. – Influence of a strong π-π interaction.

It has been surmised that a strong π-π interaction exists which would lead to a pronounced resonance in the π-π scattering (12). If the width of the resonance is sufficiently small, one could introduce this state of two pions as

(12) W. FRAZER and J. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959).

another isobar, π^* say, in the same manner as the π-$\mathcal{N}$ resonance was introduced as a $\mathcal{N}^*$.

Such a π^* gives quite good results in a statistical theory of $\bar{p}$-p annihilation [13], and it seemed therefore interesting to investigate its effect also in meson production from p-p collisions.

This computation was done on exactly the same lines as those outlined above for the $\mathcal{N}^*$ isobar. All possible reactions leading to any combination of nucleons, $\mathcal{N}^*$'s, pions and π^*'s were listed, the corresponding phase space volumes computed, and a prong analysis made, which was then combined with the analysis of the events without π^*.

For the π^* we took, as in [13]:

$$\text{mass} = 4 \text{ pion masses}\,, \qquad T = 1 \quad \text{and} \quad J = 1\,.$$

As a consequence the desintegration modes of the π^* are always unique, with a Q-value of 2μ.

$$\pi^{*+} \to \pi^+ + \pi^0\,,$$

$$\pi^{*0} \to \pi^+ + \pi^-\,,$$

$$\pi^{*-} \to \pi^0 + \pi^-\,.$$

4. – Conclusions from the purely statistical model.

The figures which can at present be compared most easily with experiment are the average pion number and the probability for different prongs; these are found in Table V and Table III. Comparing with the experimental num-

TABLE V. – *Average particle numbers* $\langle n \rangle$ *and average kinetic* $\langle \varepsilon_{\text{kin}} \rangle$ *energies (in units* 0.938 MeV) *of nucleons and pions from* 6.2 GeV p-p *collisions, averaged over all events.*

	Assuming $\mathcal{N}^*$		Assuming $\mathcal{N}^*$ and π^*	
	$\langle n \rangle$	$\langle \varepsilon_{\text{kin}} \rangle$	$\langle n \rangle$	$\langle \varepsilon_{\text{kin}} \rangle$
neutrons	0.792	289	0.794	240
π^0	0.959	385	1.23	301
nucleons (p and n)	2.00	288	2.00	240
pions (π^-, π^0, π^+)	3.00	385	3.65	302
protons	1.21	287	1.21	240
$\pi^\pm$	2.04	388	2.51	302
prongs (p and $\pi^\pm$)	3.25	350	3.72	282

[13] F. CERULUS: *Nuovo Cimento*, **14**, 827 (1959).

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bers [2] it seems that the assumption of a $\mathcal{N}^*$ (without π^*) yields the right average pion number of 3.0 and a good fit to the prong distribution, within the experimental errors.

However, the average energy of the pions, in this case, is too high. We compute 360 MeV (kinetic energy), and the measured value is 205. The discrepancy is in the other direction for the protons. We compute, for the 2-prong events, an average proton energy of 353 MeV, and the measured value is about 550.

These two facts are reflected in the degree of inelasticity K defined as the ratio of the average total energy carried off by the pions to the total available kinetic energy in the C.M. system

$$K_{\text{computed}} = 0.74 \qquad \text{(on the assumption of a } \mathcal{N}^*\text{)},$$

$$K_{\text{measured}} = 0.49 \pm 0.05$$

If we turn now to the other assumption (existence of $\mathcal{N}^*$ and π^*), it appears that the average pion number 3.65 is too high and the prong distribution gives too many 4-prong events and not enough 2-prong events.

The average pion kinetic energy of 283 comes nearer to the measured values but is still too high. The proton kinetic energy of 225 MeV gets still worse than in the no-π^* case. The degree of inelasticity we find is

$$K_{\text{computed}} = 0.76$$

and consequently neither of the two versions can explain the experiments.

In Figs. 1 to 12 are given the spectra of charged particles coming from central p-p collisions at 6.2 GeV.

In Figs. 1 to 9 are shown the spectra of p and of $\pi^\pm$ due to events with a given number of prongs; for each prong number the weighted sum of p and $\pi^\pm$ spectra is also shown (the « prong spectrum »).

Figs. 10, 11 and 12 give the p, $\pi^\pm$ and prong spectra averaged over all events. The curves give the number of particles per MeV kinetic energy, and are all normalized to unit area. The relative weights of 2, 4 or 6 prong events are shown in Table IV in the text. The continuous lines (——) refer to the hypothesis of a nucleon-pion isobar, $\mathcal{N}^*$ (corresponding to the $\frac{3}{2}$, $\frac{3}{2}$ resonance). The dotted lines (- - -) refer to the hypothesis of a $\mathcal{N}^*$ *and* of a pion-pion isobar π^* (corresponding to a π-π resonance with $E_{\text{tot}} = 4\mu$, $T = 1$ and $J = 1$).

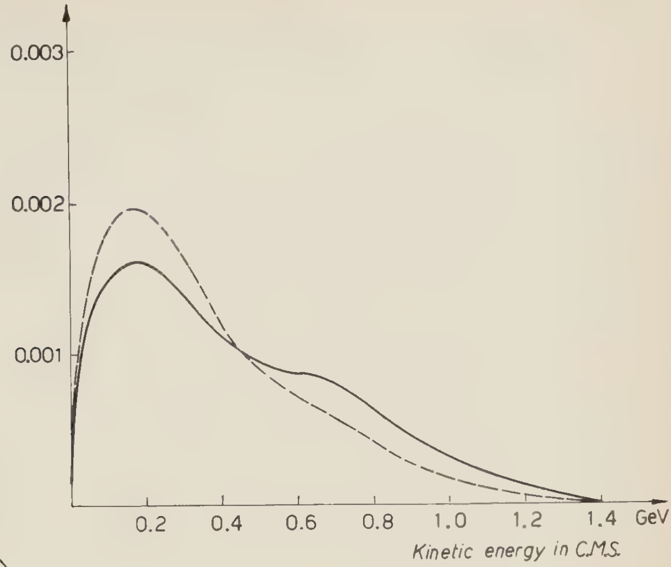

Fig. 2.

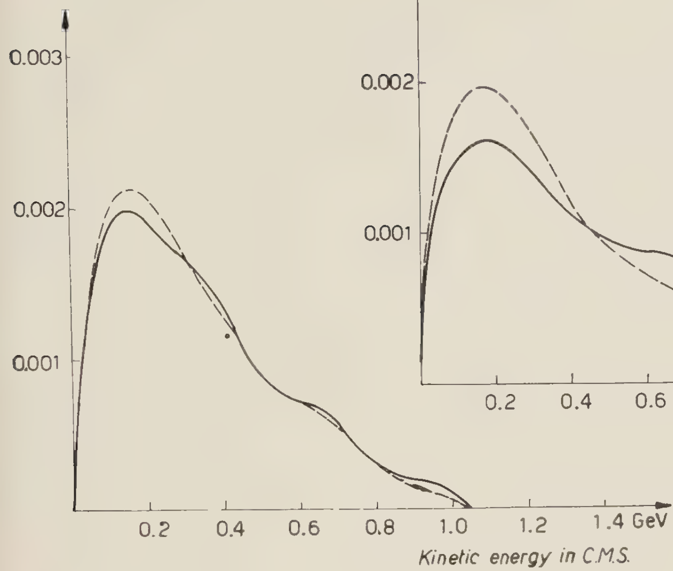

Fig. 1.

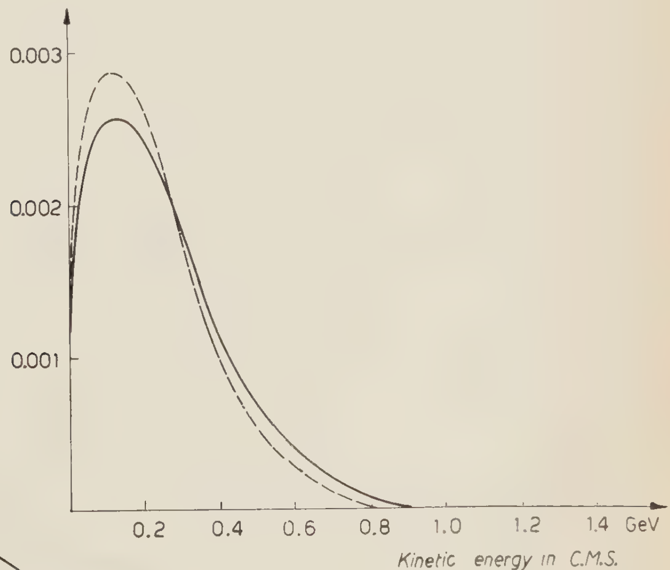

Fig. 4.

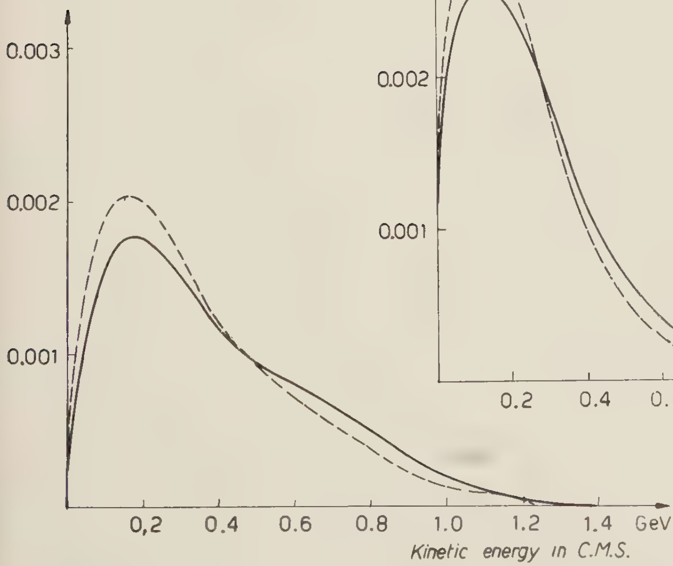

Fig. 3.

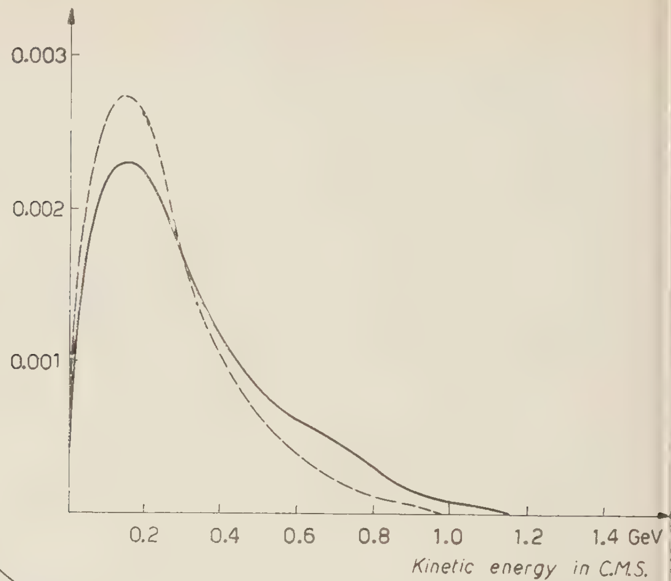

Fig. 6.

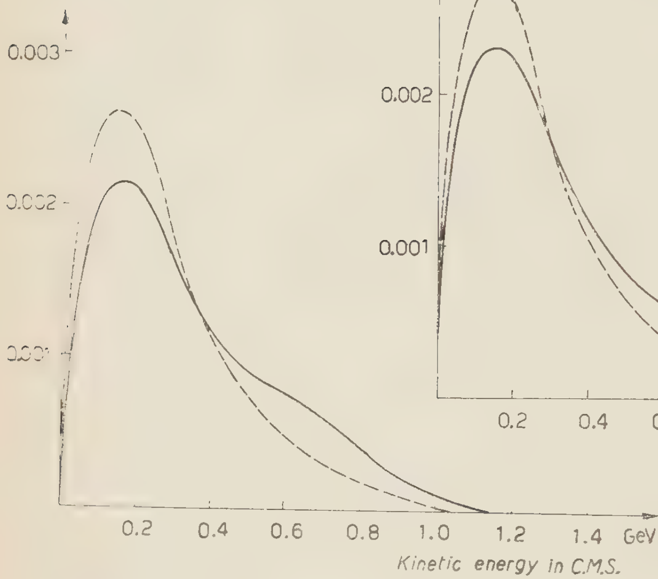

Fig. 5.

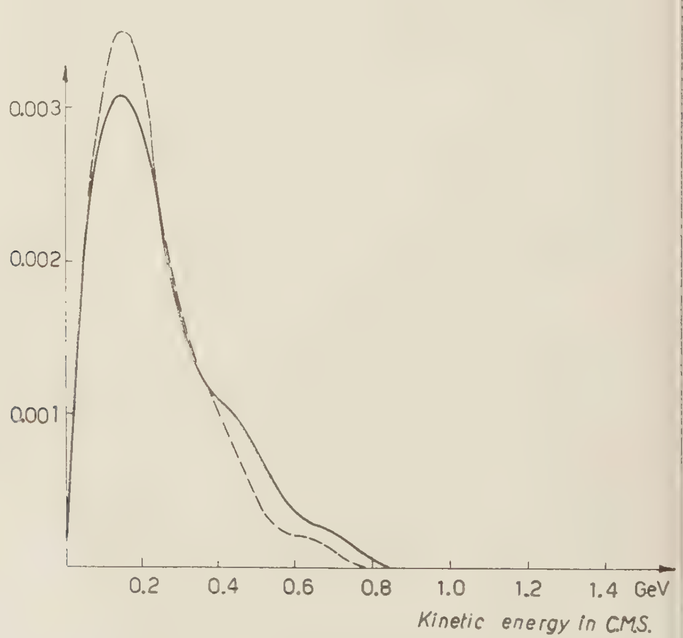

Fig. 8.

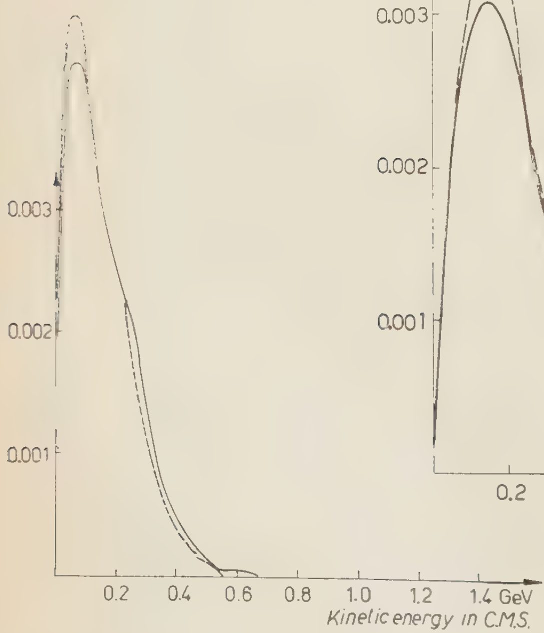

Fig. 7.

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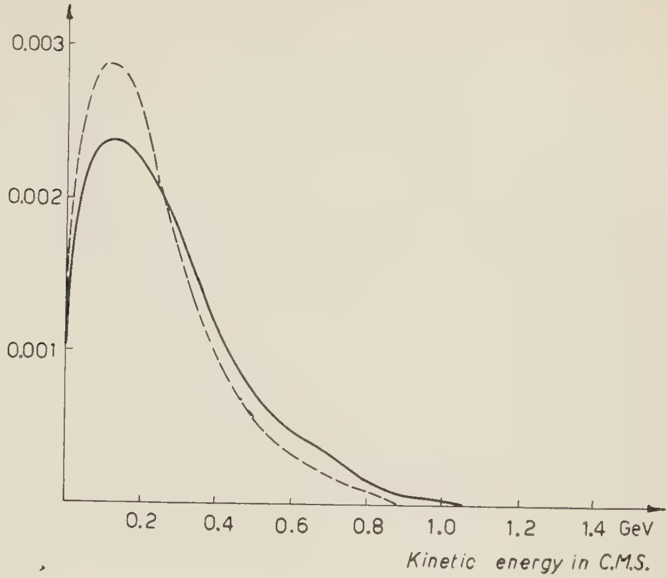

Fig. 10.

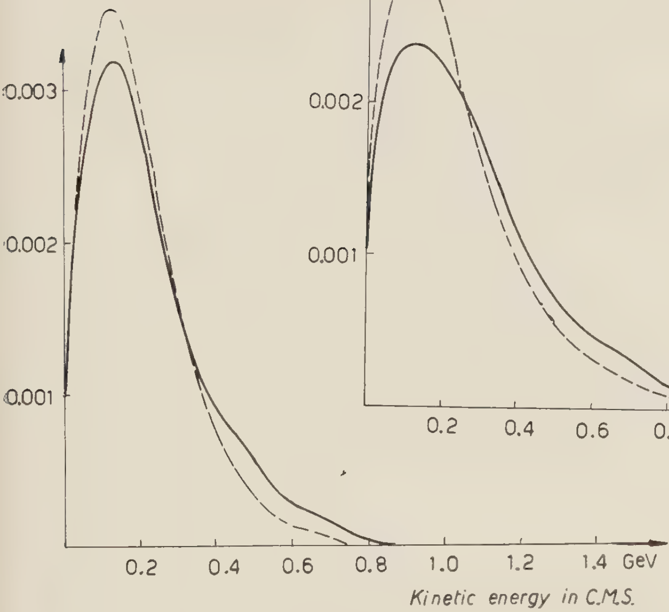

Fig. 9.

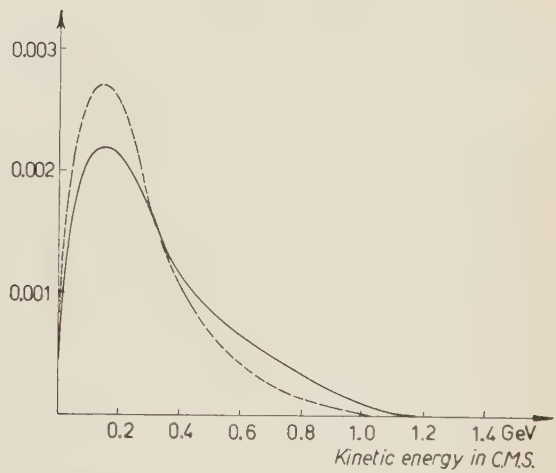

Fig. 12.

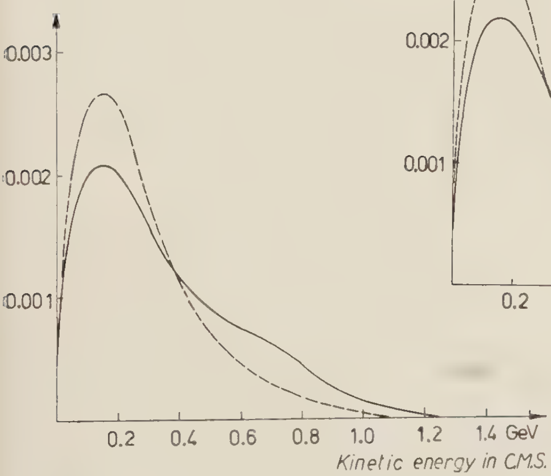

Fig. 11.

5. – The effect of peripheral collisions.

The comparison of the computed values with the measured ones strongly suggests that the assumptions of the purely statistical theory are not justified, especially the assumption of a statistical distribution of the available energy between the outgoing particles. The nucleons seem to retain on the average a substantial portion of their initial energy, and the full kinetic energy of the colliding protons is not available to the pions.

At this point one should recall that only a fraction of all collisions are central collisions, for which the statistical model could be justified. In the glancing collisions it is not possible for the perturbation—which occurs at the edge of the nucleon—to spread over the two nucleons before the bulk of them separates again. One would think much more, for such glancing collisions, of a so-called two-centre model in which the two nucleons are excited and which, after separating, radiate off mesons. Such a model yields automatically a high elasticity and a low multiplicity, and is, so to say, complementary to the purely statistical theory which should be able to treat only the central collision adequately.

Peripheral collisions, which yield only one or two mesons, and involve mostly the meson cloud of the nucleons should most adequately be treated by meson theory. What we shall do here is only to use a crude model, so as to obtain a qualitative estimate of the peculiarities of such collisions (*).

In such a two-centre model one has to assume a law for the excitation of the two centres. In the present case of glancing collisions, where only a small part of the available energy could be converted to excitation energy of the nucleon, this excitation should be governed again entirely by the predominance of the $\frac{3}{2}$-$\frac{3}{2}$ resonance at low relative π-$\mathcal{N}$ energies.

We shall make the working hypothesis that this excitation consists mainly of exciting one or both nucleons to isobars $\mathcal{N}^*$, which then fly off along approximately the same direction as the incoming protons.

We shall simplify still further, and consider all collisions to be of three kinds:

a) « central collisions » where the statistical theory applies and the whole kinetic energy is available for particle production;

(*) *Note added in proof.* – V. S. BARAŠENKOV, V. M. MALTSEV and L. K. MIKUL have also considered peripheral p-p collisions (*Nucl. Phys.*, **13**, 583 (1959) and JINR preprint, p. 433 (Nov. 1959)). They treat these as collisions between a nucleon and a pion from the second nucleon's cloud, and use then a statistical theory of π-$\mathcal{N}$ collisions to compute the peripheral collisions.

b) « peripheral collision », where one of the nucleons is excited into an isobar;

c) « peripheral collision » where both nucleons are excited.

We repeat that we consider this whole argument only as a leading to a working hypothesis which will have to be justified by its consequences.

According to this model all quantities such as energy spectra, prong distributions etc., should be calculated by taking a weighted average of the quantities predicted by the statistical model and those predicted by the « two centre » model.

One could then by comparison with some experimental numbers—the prong distribution say—derive the ratios of central collision to peripheral $\mathcal{N}+\mathcal{N}^*$ to peripheral $\mathcal{N}^*+\mathcal{N}^*$ collision.

With those numbers one should then be able, if the working hypothesis is any good, to derive the inelasticity and the spectra, and even to state something about angular distribution.

6. – Amount of peripheral collision.

The reactions $\mathcal{N}+\mathcal{N}^*$ and $\mathcal{N}^*+\mathcal{N}^*$ lead to the prong distribution given in Table VI.

TABLE VI. – *Prong distribution in peripheral collisions.*

		$\mathcal{N}+\mathcal{N}^*$	$2\mathcal{N}^*$
2 prongs	p p	0.1666	0.1777
	n p	0.8333	0.5777
	n n	—	0.0444
4 prongs	p p	—	0.2

If we call

p_0 the probability to have a central collision

p_1 the probability to have a peripheral collision of type $\mathcal{N}+\mathcal{N}^*$

p_2 the probability to have a peripheral collision of type $2\mathcal{N}^*$

and f_2, f_4, f_6 the experimentally measured frequencies of 2, 4 or 6-prong events,

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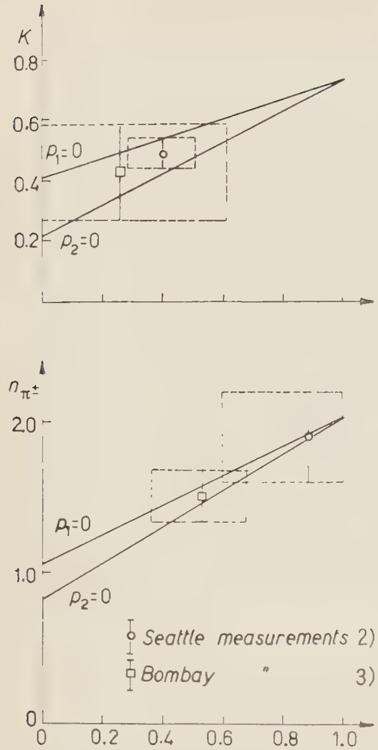

we have then the following equations for p_0, p_1 and p_2:

$$\text{(3a)}\qquad \left.\begin{matrix}0.434\\0.255\end{matrix}\right\}\, p_0+p_1+0.8p_2=f_2\,,$$

$$\text{(3b)}\qquad \left.\begin{matrix}0.526\\0.635\end{matrix}\right\}\, p_0+0.2p_2=f_4\,,$$

$$\text{(3c)}\qquad \left.\begin{matrix}0.040\\0.110\end{matrix}\right\}\, p_0=f_6\,,$$

Fig. 13. – The continuous lines show the dependence of the inelasticity K (defined on p. 1054) and of the number of charged pions on the ratio p_0 of the number of central collisions to the number of all inelastic collisions. The line « $p_2=0$ » corresponds to peripheral collisions in which only 1 isobar $\mathcal{N}^*$ is produced, $p_1=0$ to peripheral collisions producing only $2\mathcal{N}^*$ (two extreme cases; a reasonable line should lie between the two) the model assumes that peripheral collisions lead only to those two channels. Points, representing the measured inelasticity and $\pi^\pm$ number have been drawn on the intersection of a horizontal line through the measured value with the line corresponding to $p_1=p_2$ thus giving a value for p_0. The experimental uncertainty on K and $n_{\pi^\pm}$ (vertical segments) gives rise — assuming the model is correct — to an uncertainty for p_0 (horizontal segments).

TABLE VII. – *Average number $\langle n\rangle$ and average kinetic energy $\langle\varepsilon\rangle$ (in units $\sim$ 0.938 MeV) of particles from glancing collision.*

Channel	$\langle n\rangle_{p,\,\text{direct}}$	$\langle\varepsilon\rangle_{p,\,\text{direct}}$	$\langle n\rangle_{p,\,\text{decay}}$	$\langle\varepsilon\rangle_{p,\,\text{decay}}$	$\langle n\rangle_{\pi'}$	$\langle\varepsilon\rangle_{\pi'}$
$\mathcal{N}+\mathcal{N}^*$	0.25	975	0.917	725	0.8333	325
$\mathcal{N}^*+\mathcal{N}^*$	—	—	1.333	650	1.0667	300

where the upper coefficients in the first column refer to the $\mathcal{N}^*$ statistical theory, the lower ones to the $(\pi^*+\mathcal{N}^*)$ theory with $\Omega=L\Omega_0$.

The inelasticity and the average number of charged particles can now be computed, from Table V and Table VII.

The probabilities for 2, 4, 6-prong events, the inelasticity K and the number of charged particles C are plotted as functions of p_0 in Figs. 13-16.

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These figures represent the equations (3). As a function of p_0 various observables are plotted, p_1 or p_2 set to zero as a parameter. Then the measured values of the observables are inserted and the values of p_0 compatible with the experimental error are indicated by dotted lines. One sees that the ratio p_1/p_2 does not have much effect on the result, but that the important quantity is p_0, *i.e.* the fraction of central collision.

The published measurements of f_2, f_4, f_6 have rather wide errors, and it will be better to try to fit first the inelasticity and the prong number, which are better measured [2-4].

It appears at once that, due to the effect of the peripheral collisions, it is pos-

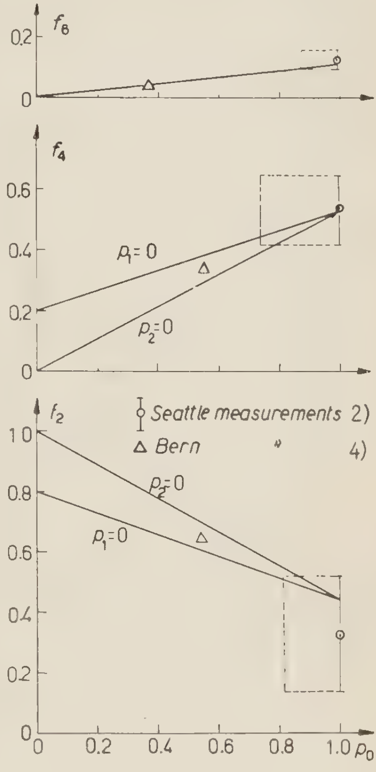

Fig. 14. – Displays in the same way as in the previous figure the relative frequencies of 2, 4 and 6-prong events. (For the Bern measurements no uncertainties were stated). Both Figs. 13 and 14 are drawn assuming an $\mathcal{N}^*$ in the central collision, but no π^*.

TABLE VIII. – *Average particle numbers* $\langle n\rangle$ *and average kinetic energies* $\langle\varepsilon\rangle$ (*in units* 0.938 MeV) *of nucleons and pions from* 6.2 GeV p-p *collisions for* $\Omega=(2M/E)\Omega_0$, *and for equal numbers of central and peripheral collisions* ($p_0=0.5$, $p_1=p_2=0.25$).

Type of event			Assuming $\mathcal{N}^*$				Assuming $\mathcal{N}^*$ and π^*			
number of prongs	number of p	number of $\pi^\pm$	$\langle n\rangle_p$	$\langle\varepsilon\rangle_p$	$\langle n\rangle_{\pi^\pm}$	$\langle\varepsilon\rangle_{\pi^\pm}$	$\langle n\rangle_p$	$\langle\varepsilon\rangle_p$	$\langle n\rangle_{\pi^\pm}$	$\langle\varepsilon\rangle_{\pi^\pm}$
2	2	0	0.230	573	—	—	0.188	721	—	—
	1	1	0.449	601	0.449	367	0.435	639	0.435	336
	0	2	—	—	0.093	432	—	—	0.083	349
4	2	2	0.320	395	0.320	367	0.329	381	0.319	334
	1	3			as in Tab. IV					
	etc.				»					
average over all events			1.23	502	1.50	364	1.23	479	1.73	304

sible to fit C and K simultaneously, with or without the π^*-hypothesis, indicating a p_0 around 0.5.

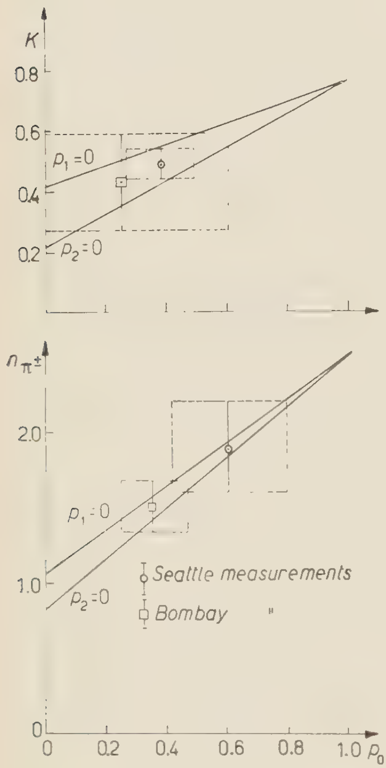

Fig. 15.

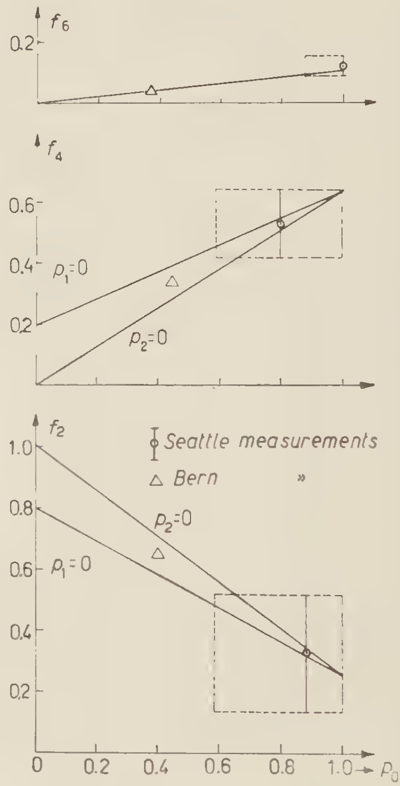

Fig. 16.

Figs. 15 and 16. – Are like 13 and 14, but now assuming $\mathcal{N}^*$ and π^* in the central collisions.

With this value of p_0 more or less fixed, it appears then that f_2 and f_4 can be fitted within the limits of error of the Seattle measurements ([2]), only if the π^* is assumed. With the Bern measurement ([4]), the fit can be achieved with and without the π^* hypothesis.

Further information is contained in the spectra. In Table VIII are shown the calculated mean energies of protons and pions, under the assumption $p_0 = 0.5$, $p_1 = p_2 = 0.25$.

This is to be compared with the experimental numbers given in Table IX.

It seems, from a comparison of the experiments and the theory, that the present model, with perhaps a more careful choice of the values of Ω and p_0, could very well predict all such features of the production process that are

TABLE IX. – *Average numbers and kinetic energies of protons and charged pions (experimental). Energy unit of* 0.938 MeV.

	mean prong number	mean ε_p	mean $n_{\pi\pm}$	mean $\varepsilon_{\pi\pm}$
Seattle measurements (2)	—	550 ± 1	$1.9\ \pm0.3$	252 ± 21
Bombay measurements (3)	—	636 ± 96	1.51 ± 0.18	309 ± 48
Bern measurements (4)	2.8	—	—	—

concerned with quantities averaged over all angles (*i.e.* mean particle numbers and spectra).

7. – Angular distributions.

In deriving the statistical model Hagedorn [14] has shown that it makes *sensu stricto* no prediction about angular distribution, but is only expected to yield correctly quantities averaged over angles.

The model we use here does, however, make definite predictions about the angular distribution of the particles from peripheral collisions.

If we assume that the isobars move along the axis defined by the incoming protons before they decay isotropically with respect to their own rest-system, the resulting p and π will have a definite correlation between their energy and angle in the C.M.S. It turns out that this correlation is nearly the same in the two cases, where either one or two $\mathcal{N}^*$ are produced in a collision.

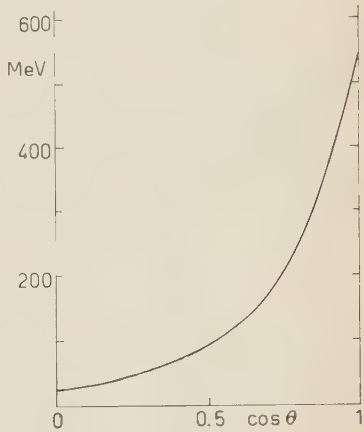

Fig. 17. – Shows the relation between the energy of a π from an $\mathcal{N}^*$ decay and its angle with the collision line, assuming the outflying $\mathcal{N}^*$ has nearly the same direction as the incoming p.

In Fig. 17 this correlation is given for the case, where the $\mathcal{N}^*$ has the direction $\theta = 0°$. As with equal probability the $\mathcal{N}^*$ will have the direction $\theta = 180°$ this relation has to be reflected with respect to the plane $\theta = 90°$.

While a resulting nucleon will move in a direction of at most 11° with respect to the incoming nucleon, the resulting pion will have a less peaked angular distribution as shown in Figs. 18 and 19.

([14]) R. HAGEDORN: *Nuovo Cimento*, **15**, 434 (1960).

We could now make the additional assumption that the central collisions yield approximately isotropic distributions.

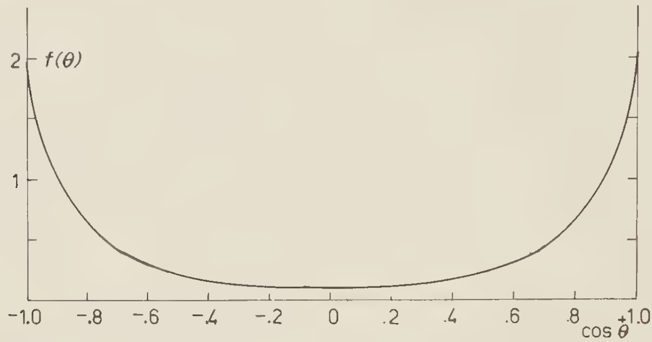

Fig. 18. – Assuming an isotropic decay of the $\mathcal{N}^*$ in its own rest frame one computes an angular distribution of the π-directions with respect to the $\mathcal{N}^*$ direction, which is — according to the model — quite close to the direction of the colliding p. In this figure one assumes production of a nucleon and an isobar $\int f(\theta)\,d\cos\theta = 0.833$.

This isotropy is suggested by the experimental distribution of 6-prong events [2] which, at least in our model, are entirely due to central collisions.

Combining this isotropic distribution with the computed anisotropy of the peripheral collisions one is able to predict the angular distribution of product particles as well as the average dependence of the energy spectra.

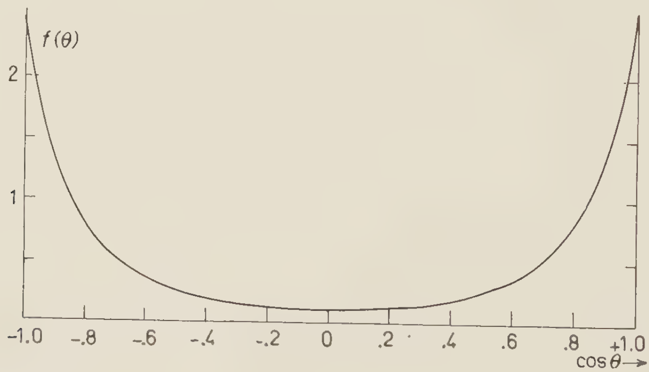

Fig. 19. – Same as Fig. 18, but for production of two isobars $\int f(\theta)\,d\cos\theta = 1.065$.

If one considers a fixed angle, in addition to the continuous spectrum originating from central collisions a line spectrum will arise as a consequence of the angle-energy-correlation shown in Fig. 17. Of course this line spectrum

will have a large width due to the energy width of the pion-nucleon isobar. As an example we show in Fig. 20 the combined energy spectrum for $\theta = 0°$ (of $\theta = 180°$).

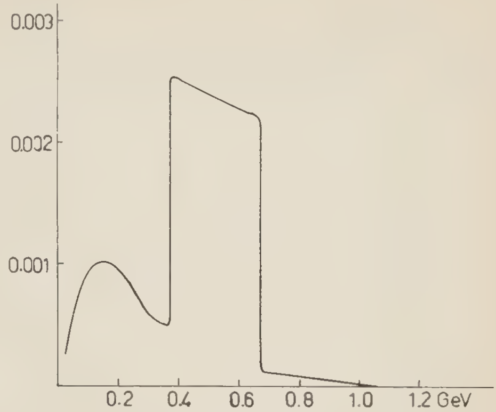

Fig. 20. – This spectrum for charged π's is composed of the continuous spectrum from central collisions and the « line »-spectrum from peripheral collisions for $\theta=0°$. The relative number of charged π's from central collisions is only 36% at this angle. (Assuming $p_0 = 0.5$, $p_1 = p_2 = 0.25$).

In comparing with experiment one should bear in mind the two special assumptions made in deriving the figures of this section:

a) isotropy of the products of central collisions;

b) direction ot the two centres in peripheral collisions = direction of incoming protons.

These assumptions are of a less convincing nature than the others made in this theory and should be checked experimentally. The study of angular correlations (cf. next section) in certain types of events allows to study peripheral collisions and to check the isobar model independently of the assumptions just mentioned.

8. – Angular correlation.

Charge analysis of peripheral collisions shows that the channel with the doubly-charged isobar will be relatively probable. Because this isobar is decaying into a proton and a π^+ one will find an angular correlation between these particles. For those two-prong events resulting in proton and π^+ a detailed analysis gives

$$g(\theta)\,\mathrm{d}\cos\theta = \tag{4}$$

$$= \frac{\{0.08p_0 + [0.75p_1 + 0.4p_2]\,C(\theta) + [0.083p_1 + 0.177p_2]\,C(\pi-\theta)\}\,\mathrm{d}\cos\theta}{0.16p_0 + 0.833p_1 + 0.577p_2},$$

where $g(\theta)$ is the probability density for the angle between the π^+ and the direction of the isobar from which it originates. Because the angle between the isobar and the proton into which it decays is at most 11° the above formula is approximately correct if θ is interpreted as the angle between p and π^+.

The function $C(\theta)$ is given in Fig. 22 while $g(\theta)$ is shown in Fig. 21 for $p_0 = \frac{1}{2}$, $p_1 = p_2 = \frac{1}{4}$. In deriving (4) we assumed that those two-prong events which originate in central collisions will exhibit no angular correlation, as

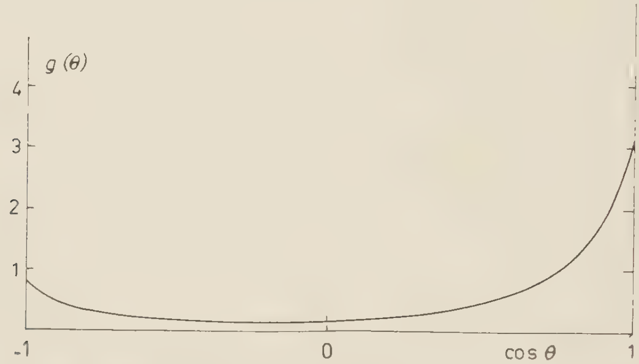

Fig. 21. – This angular correlation function $g(\theta)$ gives the relative probability of finding a π between $\cos\theta$ and $\cos\theta + d\cos\theta$, θ being the angle between the π^+ and the proton in two-prong events.

they are produced mainly by other channels. It is interesting to note that this angular correlation is independent of the assumption that the isobars fly away along the collision line. This is due to the fact that in two centre col-

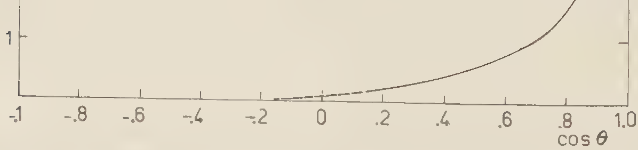

Fig. 22. – The function $C(\theta)$ is obtained by calculating the probability of finding the π between $\cos\theta$ and $\cos\theta + d\cos\theta$ in the C.M. system, assuming that the $\mathcal{N}^*$ decays isotropically with respect to its own rest system. The direction of the $\mathcal{N}^*$ is 0°. As the velocity of the CMS is comparable within the velocity of the π in the $\mathcal{N}^*$ system most π's fly into the forward direction.

lisions both centres must have opposite directions. Even if it will not be possible to distinguish between protons and pions one should find a similar

angular correlation between the couples of charged particles in two prong events. The main difference will be an increase of the function $g(\theta)$ between $\cos\theta = -1$ and $\cos\theta = -0.95$ due to couples of proton tracks.

* * *

The author express their gratitude to Dr. R. HAGEDORN for numerous discussions, valuable comments and the use of his computer programmes.

Experiments were discussed with Dr. WINZELER whom we thank for early communication of the Bern results.

RIASSUNTO (*)

Si propone un modello in cui le collisioni « centrali » vengono trattate secondo una teoria statistica e le collisioni « periferiche » secondo un semplice modello a due centri. La probabilità relativa dei due tipi di collisioni viene introdotta come nuovo parametro. Con questo solo parametro è possibile avere delle concordanze con i valori sperimentali dell'anelasticità, del numero medio di pioni, del rapporto fra gli eventi a due e quattro o sei rami e delle energie medie. L'introduzione della isobara π-π cambia il risultato, ma gli esperimenti attuali, per la mancanza di precisione, non possono nè confermare nè escludere questa ipotesi. Il modello predice anche le distribuzioni angolari ed alcune correlazioni angolari, come pure gli spettri di energia. Nella prima parte del presente studio si mostra che un modello puramente statistico (senza collisioni « periferiche ») non rende conto della anelasticità piuttosto bassa che si trova sperimentalmente, perchè fornisce energie dei mesoni troppo alte ed energie dei nucleoni esageratamente basse.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVI, N. 6 16 Giugno 1960

On Pauli's Transformation.

J. M. JAUCH (*)

Office of Naval Research, London Branch Office - London

(ricevuto il 30 Marzo 1960)

Summary. — It is shown that Pauli's transformation which mixes particle with antiparticle states is canonical but not unitary. It leads from one representation of the field operators to a new inequivalent one.

PAULI [1] has shown that for a spinor field ψ with vanishing mass the transformation

$$\left\{\begin{array}{l} \psi \to \psi' = \alpha\psi + i\beta\gamma_5\psi^c \\ \text{with} \\ |\alpha|^2 + |\beta|^2 = 1 \end{array}\right. \tag{1}$$

leaves the commutation rules of the field-operators invariant and is thus canonical. We use the notation of reference [2]:

$$\begin{aligned} \gamma_5 &= \gamma_0\gamma_1\gamma_2\gamma_3 \\ \{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu} \\ \psi^c &= C^*\psi^* \\ \gamma_\mu^* &= C\gamma_\mu C^{-1} \\ C^*C &= I\,. \end{aligned}$$

(*) On leave of absence from the University of Geneva.

(1) W. PAULI: *Nuovo Cimento*, 6, 204 (1957).

(2) J. M. JAUCH and F. ROHRLICH: *Theory of Photons and Electrons* (New York, 1960).

The commutation rules for the field operators are

$$\{\psi(x), \overline{\psi}(y)\} = iS(x-y) = \{\psi^c(x), \psi^c(y)\}, \tag{2}$$

$$S(x) = \partial\!\!\!/\, \Delta(x). \tag{3}$$

Although the transformation is canonical it is not necessarily unitary because of the existence of inequivalent representations of the field operators. We shall show that Pauli's transformation is not unitary except when $\alpha = 1$ or $\beta = 1$.

In order to formulate the problem precisely it is better to go from the field operators over to creation and annihilation operators. The formal relation between them is

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{k,r} \left(a_{k,r}\, u_{k,r} \exp[ikx] + b^*_{k,r}\, v_{k,r} \exp[-ikx]\right). \tag{4}$$

Here V is the quantization volume which we introduce in order to make the wave vector $\boldsymbol{k}$ discrete, u und v are the spinor amplitudes which for vanishing rest mass can be assumed identical and satisfying

$$k\!\!\!/\, u = k\!\!\!/\, v = 0. \tag{5}$$

Finally a and b are the annihilation operators for the particles and the antiparticles described by the spinor field. In order to simplify the notation we shall choose one single index r to replace the wave vector $\boldsymbol{k}$ and the spin orientation denoted in (4) also by r.

The annihilation operators satisfy a set of relations

$$\left.\begin{aligned} &\{a_r, a_s\} = \{b_r, b_s\} = \{a_r, b_s\} = 0 \\ &\{a_r, b^*_s\} = 0 \\ &\{a_r, a^*_s\} = \delta_{rs} = \{b_r, b^*_s\}. \end{aligned}\right\} \quad \text{for all } r,s \tag{6}$$

The transformation (1) when written in terms of these operators for suitable choice of phases in the definition of u takes on the form

$$\left\{\begin{aligned} a'_r &= \alpha a_r + \beta b_r \\ b'_r &= -\beta^* a_r + \alpha^* b_r \end{aligned}\right. \tag{7}$$

$$\left\{\begin{aligned} a'^*_r &= \alpha^* a^*_r + \beta^* b^*_r \\ b'^*_r &= -\beta a^*_r + \alpha b^*_r \end{aligned}\right. \tag{7*}$$

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In this form the canonical character of the transformation is especially easy to verify.

It is also seen in this form that the transformations (7) and (7*) mix the particle and the antiparticle states with coefficients α, β which are identic for all degrees of freedom. It is further seen that each degree of freedom is transformed into itself. There is no mixing of different components. For this reason the hypothetical transformation U which connects the new with the old operators through

$$\left\{\begin{aligned} a_r' &= U a_r U^{-1}, \\ b_r' &= U b_r U^{-1}. \end{aligned}\right. \tag{8}$$

would be a direct product of unitary matrices $U^{(r)}$

$$U = \prod_{r=1} U^{(r)}. \tag{9}$$

In the standard representation (2) the operators a_r and b_r are represented in an infinite direct product space of two-dimensional factor spaces one for each of the operators a_r and b_r. Such a pair requires thus for each r a four-dimensional space and the matrices $U^{(r)}$ are four-dimensional.

The whole question reduces now to the question whether the infinite direct product written formally in (9) converges in fact to a unitary operator U^*.

We introduce for each r the four base vectors (suppressing the index r).

$$\begin{aligned} \varphi_1 &\equiv \varphi_{00} \\ \varphi_2 &\equiv \varphi_{10} = a^*\varphi_{00} \\ \varphi_3 &\equiv \varphi_{01} = b^*\varphi_{00} \\ \varphi_4 &\equiv \varphi_{11} = a^*b^*\varphi_{00} \end{aligned}$$

and the corresponding transformed vectors

$$\begin{aligned} \varphi_1' &- \varphi_{00}' \\ \varphi_2' &- \varphi_{10}' = a'^*\varphi_{00}' \\ \varphi_3' &- \varphi_{01}' = b'^*\varphi_{00}' \\ \varphi_4' &- \varphi_{11}' = a'^*b'^*\varphi_{00}'. \end{aligned}$$

The state φ_{00} is the « vacuum » state satisfying

$$a\varphi_{00} = b\varphi_{00} = 0.$$

We define the matrix elements

$$U_{ik} = (\varphi_i,\, U\varphi_k)$$

and find after a short calculation

$$U_{11} = U_{44} = 1$$

$$U^*_{22} = U_{33} = \alpha$$

$$-U_{23} = U^*_{32} = \beta\,;$$

and all other components are zero.

Reintroducing now the index r for the different degrees of freedom we write formally for the general matrix element

$$U_{i_1 i_2 \dots | k_1 k_2 \dots} = \prod_{r=1}^{\infty} U^{(r)}_{i_r k_r}\,. \tag{10}$$

In order to abbreviate the writing we introduce the notation

$$\begin{cases} \{i\} = i_1 i_2 \dots \\ \{k\} = k_1 k_2 \dots \end{cases}$$

We now choose a particular set i in such a way that in the infinite sequence of i_r only a finite number are equal to 1 or 4. There must then be infinitely many equal to 2 or 3, or both. Let us assume that the index 2 occurs infinitely many times. In the infinite product we have at least one term equal to zero unless $k_r = 2$ or 3 is always paired with the index $i_r = 2$. At least one kind of pairing must occur infinitely many times, let us assume it is $i_r = 2$, $k_r = 3$. The matrix element for this set of indices is always β for each r. Thus the infinite product contains a factor consisting of all the matrix elements with this particular pairing and this factor is the infinite product $\prod \beta$. This product diverges to zero unless $\beta = 1$ and consequently $\alpha = 0$. For the other pairing $i_r = 2$, $k_r = 2$ the product vanishes unless $\beta = 0$, $\alpha = 1$.

The vanishing of this product means that for this particular choice of the column index $\{i\}$ *all* matrix elements $U\{i\}\{k\}$ vanish identically no matter what $\{k\}$. Thus a whole row of the matrix U vanishes identically. Such a matrix cannot be unitary. It violates for instance the relation

$$UU^* = I\,.$$

We mention in conclusion that it is possible to generalize the transformation in such a manner that it becomes unitary. It is merely necessary to introduce the parameters α and β as function of the index r, such that the infinite products $\prod \alpha_r$ and $\prod \beta_r$ converge to a finite number. This can be done in many different ways. All these transformations for which the infinite product converges have one feature in common, *viz.* they cannot be written as local transformations in the field operators. It is perhaps just this non-local character of such unitary transformations which make them interesting in a possible future theory.

RIASSUNTO (*)

Si mostra che la trasformazione di Pauli che mescola stati di particella con stati di antiparticella è canonica ma non unitaria. Essa porta da una rappresentazione degli operatori di campo ad un'altra non equivalente.

(*) *Traduzione a cura della Redazione.*

Pion Production and the «Isobaric» Model.

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Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

(ricevuto il 31 Marzo 1960)

Summary. — A model is proposed in order to take into account interference effects between different (33) isobaric channels in pion production. Within the one GeV energy region the qualitative features of the Lindenbaum and Sternheimer isobaric model are confirmed. At lower energies the interference modifies radically the isobaric spectra.

1. – Introduction.

Some years ago Lindenbaum and Sternheimer [1-3] have proposed a model for the pion production which allows the calculation of the final particles spectra. The problem which they try to solve is the following: in a final state of a reaction there are two particles which have a strong resonant interaction: how are modified the momenta distributions of the produced particles?

To be definite: we know that the pion-nucleon scattering is resonant in the $T=\frac{3}{2}$ and $J=\frac{3}{2}$ state. In a reaction in which the pion and the nucleon are produced in this state it seems reasonable that they prefer to stay together with a mass weighted on their elastic total cross-section. In the limit of a infinitely narrow resonance (corresponding to a bound state) the reaction is modified so that the pion and the nucleon are a single particle of definite mass. L. and S. suppose that the reaction excites the pion-nucleon system with

(1) S. J. Lindenbaum and R. B. Sternheimer: *Phys. Rev.*, **105**, 1874 (1957). (Quoted as L. and S. in the following).
(2) S. J. Lindenbaum and R. B. Sternheimer: *Phys. Rev.*, **106**, 1107 (1957).
(3) R. B. Sternheimer and S. J. Lindenbaum: *Phys. Rev.*, **109**, 1723 (1958).

mass distribution $F(m)$ so that the differential cross-section is:

$$\text{(1)} \qquad \mathrm{d}\sigma_{\text{prod}} \propto F(m)\, \sigma_s(m)\, \mathrm{d}m\,.$$

In order to identify $F(m)$ L. and S. take the limit of a very narrow resonance $\sigma_s(m) = \delta(m - m_r)$. Integrating on m one has: $\int \mathrm{d}\sigma_{\text{prod}} = F(m_r)$ which has to correspond to the reaction cross section for a particle of mass m_r. So one can interpret $F(m)$ as the cross-section for a « particle » of variable mass m. Evidently F depends on the initial total energy and on the angle of production of the « isobar ».

Let us consider the process:

$$\text{(2)} \qquad \pi^+ + \mathrm{p} \to \mathrm{p} + \pi^+ + \pi^0\,.$$

If we suppose that the final pions interact with the nucleon in the $T=\frac{3}{2}$ $J=\frac{3}{2}$ state we shall have in the « isobaric » model two contributions which are well explained by the graphs of Fig. 1, which correspond respectively to the π^+ or the π^0 resonant with the proton.

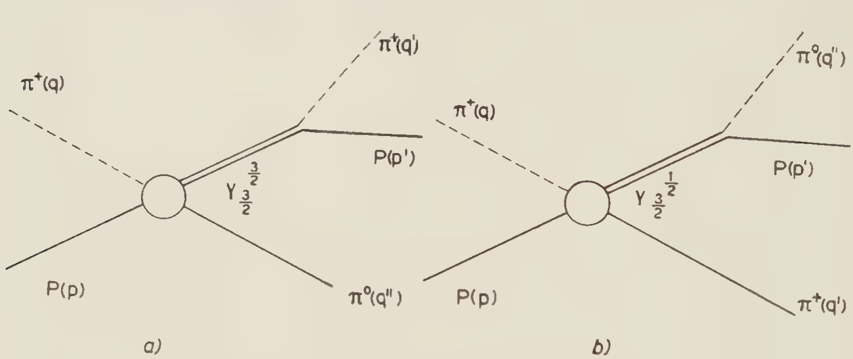

Fig. 1. – Diagrams corresponding to the two channels of the reaction: $\pi^+ + \mathrm{p} \to \mathrm{p} + \pi^+ + \pi^0$.

So L. and S. write:

$$a) \qquad \mathrm{d}\sigma_{\text{prod}}(m) \propto \alpha F(m)\, \sigma_s(m)\, \mathrm{d}m\,,$$

$$b) \qquad \mathrm{d}\sigma_{\text{prod}}(m') \propto \beta F(m')\, \sigma_s(m')\, \mathrm{d}m'\,,$$

where $m^2 = (q' + p')^2$ is the total energy in the π^+-p center of mass system and $m'^2 = (q'' + p')^2$ the total energy in the π^0-p center of mass; α and β are the isotopic spin Clebsch-Gordan coefficients. If one considers the π^0 momenta distribution, the first term corresponds to the « extra-pion » and the second to the « decay-pion » contribution in the L. and S. terminology.

In the « isobaric » model those two contributions are summed incoherently.

We wish to stress the fact that in actual experiments one observes the final state composed of the π^+, the π^0 and the proton. This means that any intermediate state contributions are to be considered coherently.

Only if the « isobar » had an infinite lifetime or if one could observe its charge in the intermediate state, then the two contributions would be incoherent due to the orthogonality of the charge states (*).

So one needs the relative phase of the two amplitudes not only the moduli of the amplitudes.

The object of the present paper is to construct an « isobaric » model for the amplitudes not for the cross-sections in order to have a qualitative evaluation of the interference between the two channels. This model has several advantages: *a*) as we have just discussed it is consistent with general principles; *b*) it sheds some light one the hypothesis underlying the « isobaric » model; *c*) it is simpler for actual computations. We shall consider only pion production from pion-nucleon reaction. We wish to stress that the same model may be used for any type of production processes like $\mathcal{N}+\mathcal{N}\to\pi+\mathcal{N}+\mathcal{N}$ and double photo-production of pions.

We have thought that it is worth-while to reconsider the « isobaric » model because in the recent time several experiments have qualitatively confirmed the main consequences of this model [4-6].

Evidently any « isobaric » model is only the first approximation to the problem of production of particles. However it is very important to have clear in mind the stringent hypothesis one has made up to now (with promising results), in order to find the key for the complete solution of the problem.

2. – The model.

Let us consider the T matrix for π-$\mathcal{N}$ scattering in the $J=\frac{3}{2}$, $T=\frac{3}{2}$ state (Y). It is well known that the approximated solution of the dispersion relations

$$(3)\qquad T_{33}=\frac{1}{q}\exp\left[i\delta_{33}\right]\sin\delta_{33}=\frac{\Gamma(m)}{(m_r-m)-i\Gamma(m)q}\,;\quad \Gamma(m)=\frac{\gamma q^2}{\omega^*}\,;\quad \gamma=\frac{4}{3}f^2\omega_r^*\,,$$

(*) One may hope to produce in the future « isobars » with higher velocities and hence sufficiently long lifetimes to be observed.

(4) V. Alles Borelli, S. Bergia, E. Perez Ferreira and P. Waloschek: *Nuovo Cimento*, **14**, 211 (1959).

(5) I. Derado and N. Schmitz: to be published in *Phys. Rev.*

(6) I. Derado, G. Lütjens and N. Schmitz: *Ann. d. Phys.*, **4**, 103 (1959).

where q is the c.m. momentum, $m^2=(q+p)^2$ is the total energy in the c.m. and $\omega^*=m-M$ (M nucleon mass), is in agreement with the presence of the resonance in this state. ($\omega_r^*\sim 2.1\,\mu$ and $\gamma\sim 0.22\,(1/\mu)$.)

An important feature of (3) is that one can consider the resonance as due to a pole in the unphysical sheet of m.

We suppose that any matrix element in which a pion and a nucleon are created in the (33) state has the same pole in the right variable (the π-$\mathcal{N}$ total energy in their c.m. system) as the scattering matrix element; this means that we factorize the denominator D.

Any other dependence on m is sufficiently weak to be neglected or to be taken at the resonance value.

These hypotheses are the generalization of what one finds in the photoproduction of a single pion and for pion production using the static model.

The presence of this pole is quite natural if one considers the perturbation theory of a stable Y particle. In Appendix I we shall show the equivalence, with some modifications, of our model to the L. and S. model if only one of the two amplitudes is considered.

We select from the pion production those contributions in which the final nucleon and one of the pions are in the (33) state.

From our hypotheses the matrix element corresponding to the process a) has the pole in the variable $m^2=(q'+p')^2$ and the matrix element for the process b) has the pole in the variable $m'^2=(q''+p')^2$; in formulae:

$$T_{\text{prod}}=\alpha\sqrt{\Gamma(m)}\,\frac{R(W,\Theta',\{\vartheta'',\varphi''\})}{D(m)}+\beta\sqrt{\Gamma(m')}\,\frac{R(W,\Theta'',\{\vartheta',\varphi'\})}{D(m')}\,,\tag{4}$$

where W is the total initial energy in the c.m. Θ', (Θ'') are the angles in which the «isobars» are produced, ϑ'', φ'' (ϑ', φ') are the decay pion's angles in the system in which the «isobars» are at rest.

We have to explain for which reason we have put the factor $\sqrt{\Gamma(m)}$ in (4): (3) and (4) have a simple intuitive meaning; (3) could be considered a second order perturbation of the π-$\mathcal{N}$ scattering through a «particle» Y of complex propagator $D(m)$, where $\sqrt{\Gamma(m)}$ is the π, $\mathcal{N}$, Y coupling.

(4) may be considered a perturbative calculation of the production, in which a phenomenological interaction for the production of the Y is taken.

We can give another justification of the factor $\sqrt{\Gamma(m)}$; we have seen that, in order to identify F in their formulae, L. and S. have to suppose that $\sigma_s(m)$ approaches, in the limit of a very narrow resonance, to $\delta(m-m_r)$. From (3), integrating over m one has:

$$\int\limits_{-\infty}^{\infty}\sigma_s(m)\,\mathrm{d}m=\int\mathrm{d}m\,\frac{\Gamma^2(m)}{(m_r-m)^2+\Gamma^2(m)q^2}\propto\frac{\Gamma(m_r)}{q_r}\,,$$

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so $\sigma_s(m)$ is not the proper δ-function, but one needs:

$$\frac{q_r}{\Gamma(m)}\,\sigma_s(m) \simeq q\,\frac{\Gamma(m)}{|D|^2}\,.$$

This is the reason to take $\sqrt{\Gamma(m)}$ in (3). The extra q comes from the final state density, as we shall show in Appendix I.

The factor $q/\Gamma(m) = \omega^*/q\gamma$ which multiplies the cross-section in the « isobaric » model is such that does not modify the L. and S. spectra appreciably. In a recent paper SANDS (7) has noted the existence of this factor from different arguments. If one has competing resonances in the final state (as for example π, π resonance) the effect on the production is smaller for larger γ, evidently supposing all other factors comparable.

The ϑ', φ' (ϑ'', φ'') dependence in (4) is given once one knows the « isobar's » polarization. So it is necessary for actual computation to know the mechanism of production, *i.e.* the Θ' and Θ'' dependence. The production process is not sufficiently well known experimentally to make any feasible hypothesis.

So as L. and S. we shall perform the computations in the case of isotropic production and decay of the « isobars » in order to have a qualitative calculation of the interference.

Evidently the isotropic decay is a very crude hypothesis and it would be right only for scalar particles.

For the squares of the two matrix elements, which are equivalent to the « extra » and « decay » pion contribution of L. and S., we have calculated the right decay distribution in the hypothesis of s wave production. It turns out to be a very small correction of the order of few per cent.

We can write the matrix element (4) as:

$$T_{\text{prod}} = f(w)\left\{\alpha\,\frac{\sqrt{\Gamma(m)}}{D(m)} + \beta\,\frac{\sqrt{\Gamma(m')}}{D(m')}\right\}. \tag{5}$$

The differential cross-section is given by (see Appendix II for the coefficient α, β)

$$\mathrm{d}\sigma_{\text{prod}} \propto \frac{1}{I\omega E}\,|T_{\text{prod}}|^2\,\frac{\mathrm{d}^3p'}{E'}\,\frac{\mathrm{d}^3q'}{\omega'}\,\frac{\mathrm{d}^3q''}{\omega''}\,\delta^4(q+p-p'-q'-q'')\,.$$

The momentum distribution for the π^+ is given by:

$$\frac{\mathrm{d}\sigma_{\text{prod}}}{\mathrm{d}q'} \propto \frac{1}{I\omega E}\,\frac{q'^2}{\omega'}\int |T_{\text{prod}}|^2\,\frac{q''^2}{E'\omega''}\,\frac{\mathrm{d}q''}{\mathrm{d}E_f}\,\mathrm{d}\Omega_{q'}\,\mathrm{d}\Omega_{q''}\,,$$

where the integrations are performed over $\Omega_{q'}$ and $\Omega_{q''}$.

---

(7) M. SANDS: California Institute of Technology, June 27, 1958 (preprint).

Evidently the variables contained in (5) are to be transformed in the general c.m. system.

One can calculate the final particles momenta distributions but not the total rate of production because of the unknown function of the total initial energy.

3. – Results.

In Fig. 2 and following we show as an example the modification of the L. S.

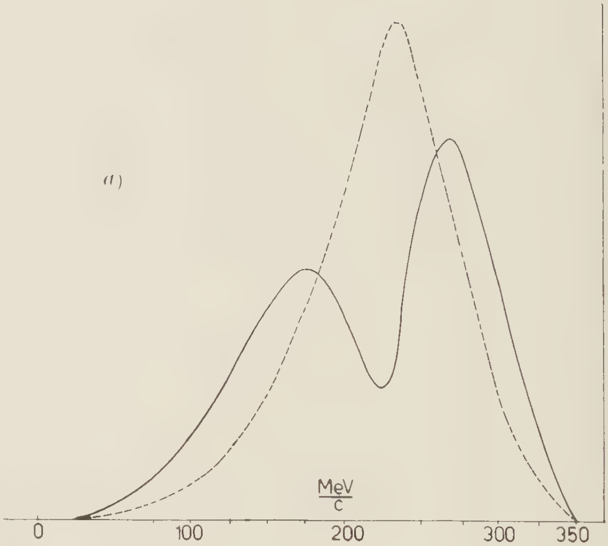


Fig. 2. – π^0 spectrum from the reaction

$$\pi^+ + p \to p + \pi^+ + \pi^0$$

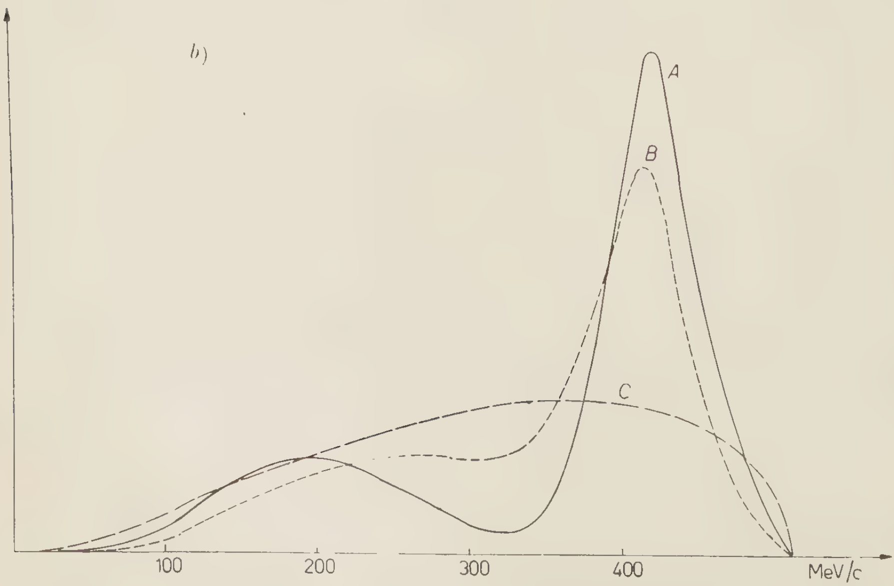
(π^+ from

$\pi^- + p \to \pi^+ + \pi^-$; $T = \frac{3}{2}$).

A) with interference; *B*) without interference; *C*) statistical.

a) 600 MeV lab. K. E. of the incident pion;

b) 950 MeV lab. K. E. of the incident pion.



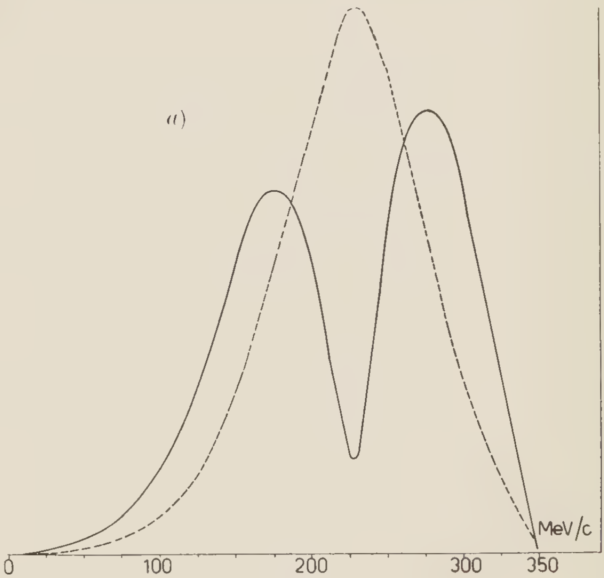
spectra due to the presence of the interference term. The calculations were done at 600 and 960 MeV for the kinetic energy of the incident pion in the lab. system (*).

As one sees the interference effects are visible in the $T=\frac{3}{2}$ state because the extra and decay pion contributions are of the same order.

At the lowest energy value the prediction of our model and the L. S. one are qualitatively different.

We suggest experiments of production by positive pions in order to check our results.

At the highest energy value the modifications are smaller due to the separa-



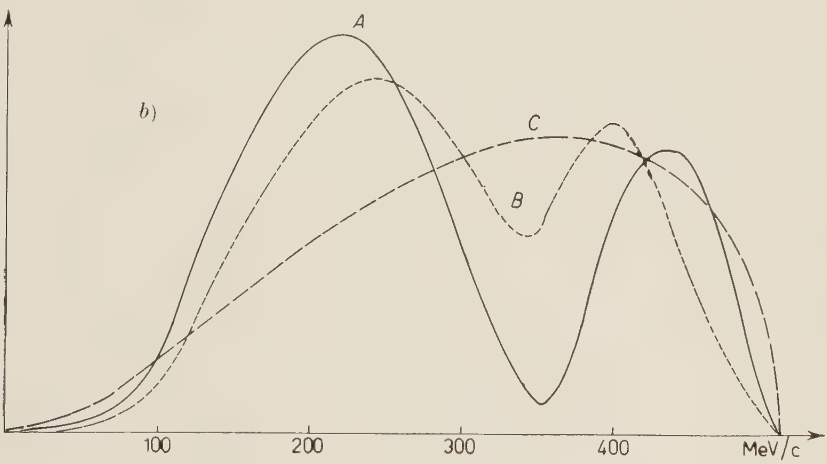


Fig. 3. – π^+ spectrum from the reaction

$$\pi^+ + p \to p + \pi^+ + \pi^0$$

(π^- from

$$\pi^- + p \to n + \pi^+ + \pi^- \;;\quad T = \tfrac{3}{2}) .$$

A) with interference; B) without interference; C) statistical.

a) 600 MeV lab. K. E. of the incident pion;

b) 950 MeV lab. K. E. of the incident pion.

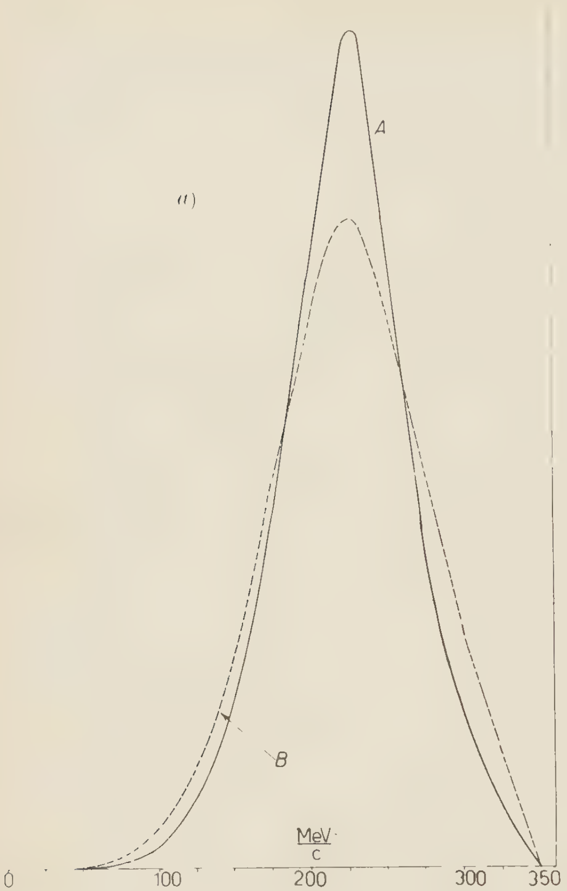


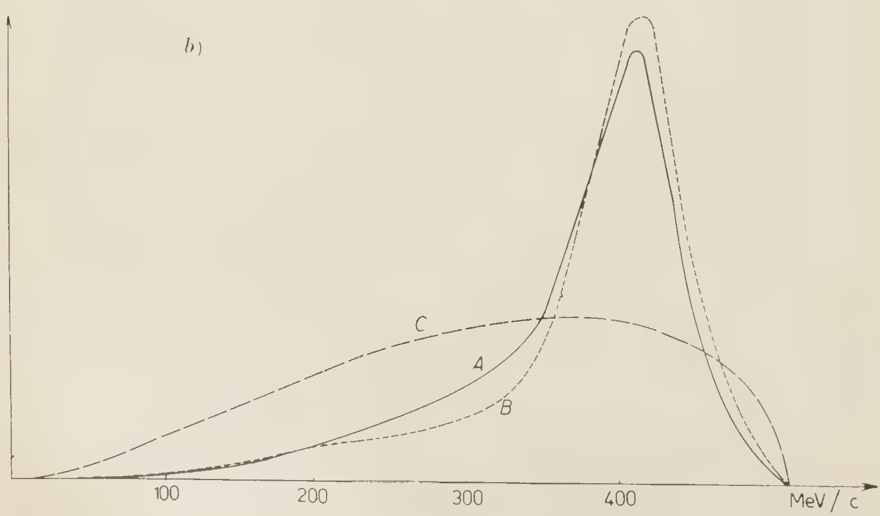
Fig. 4. – π^+ spectrum from the reaction

$$\pi^- + p \to n + \pi^+ + \pi^- ; \quad T = \tfrac{1}{2} .$$

A) with interference; *B*) without interference; *C*) statistical.

a) 600 MeV lab. K. E. of the incident pion;

b) 950 MeV lab. K. E. of the incident pion.



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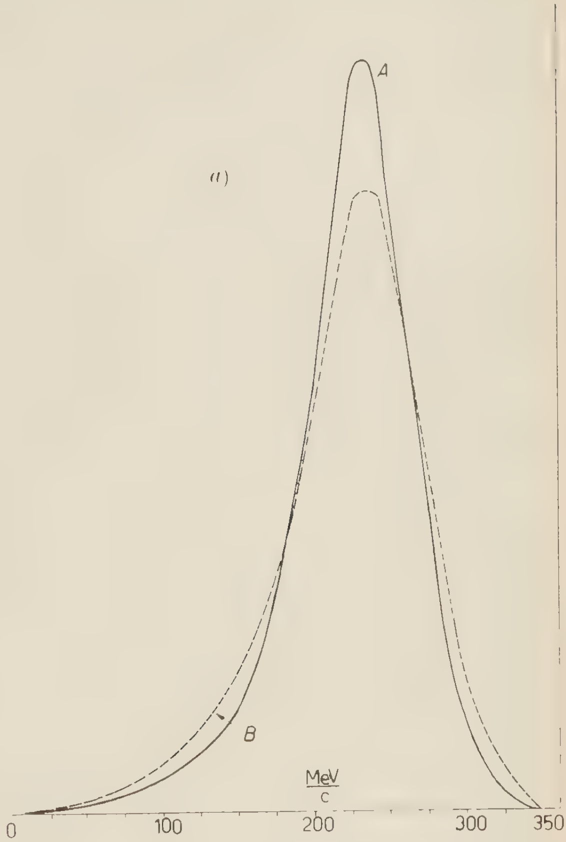


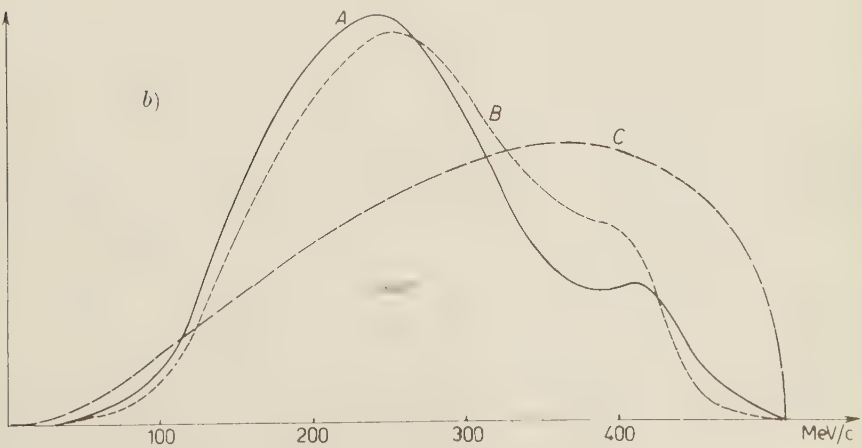
Fig. 5. – π^- spectrum from the reaction

$$\pi^- + p \to n + \pi^+ + \pi^- ; \quad T = \tfrac{1}{2} .$$

A) with interference; *B*) without interference; *C*) statistical.

a) 600 MeV lab. K. E. of the incident pion;

b) 950 MeV lab. K. E. of the incident pion.



tion of the two contributions and practically the qualitative predictions of the two models are the same.

We have clear in mind that the interference effect we have taken into account and calculated with a particular model, is only one of the many possible effects which may modify the L. S. spectra. We hope that our model which is a consistent formulation of the isobaric idea, may explain the qualitative deviation from the L. S. spectra.

In any case we wish to emphasize the fact that the two models, which have the same high energy behaviour, give different results at low energy. This makes clear the danger to look for experimental confirmation of the L. S. model at low energy pion production.

* * *

We wish to thank Profs. B. FERRETTI, S. FUBINI and G. PUPPI for helpful discussions.

APPENDIX I

We wish to show the equivalence of our model and the L. and S. one for the amplitude b (see Fig. 1).

We write the production partial cross section:

$$\text{(A.1)}\qquad \mathrm{d}\sigma_{\text{prod}} \propto \frac{1}{I\omega E}|T_a|^2 \frac{\mathrm{d}^3q'}{\omega'}\frac{\mathrm{d}^3q''}{\omega''}\frac{\mathrm{d}^3p'}{E'}\,\delta^4(q+p-p'-q'-q'')\,.$$

In order to separate the appropriate variables we parametrize in the following manner:

$$\text{(A.2)}\qquad \begin{cases} \mathrm{d}\sigma_{\text{prod}} \propto \dfrac{1}{I\omega E}\dfrac{\mathrm{d}^3q'}{\omega'}\,\delta^4(p+q-q'-t)\,\mathrm{d}^4t\,\delta(t^2-\boldsymbol{m}^2)\,\mathrm{d}\boldsymbol{m}^2B\,, \\[2ex] B=\dfrac{\mathrm{d}^3q''}{\omega''}\dfrac{\mathrm{d}^3p'}{E'}\,\delta^4(t-p'-q'')\,|T_a|^2\,. \end{cases}$$

The separation (A.2) is relativistically invariant.

We calculate B in the isobar rest system, *i.e.* $p'+q''=0$ after having performed the integrations which eliminate the δ functions. From our hypothesis:

$$\text{(A.3)}\qquad \begin{cases} T_a=\dfrac{\sqrt{\Gamma(m)}}{D(m)}\,R(W,\Theta',\{\vartheta'',\varphi''\})\,, \\[2ex] B\;=\mathrm{d}\Omega_{\bar q}\dfrac{\bar q}{m}\dfrac{\Gamma(m)}{D(m)^2}\,|R(W,\Theta',\{\vartheta'',\varphi''\})|^2\,, \end{cases}$$

where $\bar q$ is the π^0 momentum in the Y rest system.

Performing in (A.3) the angular integration one gets:

$$B \propto F(W, \Theta') \frac{\bar{q}}{m\Gamma(m)} \sigma_s(m) . \tag{A.4}$$

We substitute (A.4) in (A.2):

$$\mathrm{d}\sigma_{\text{prod}} \propto \frac{1}{I\omega E} \frac{\mathrm{d}^3 q'}{\omega'} \frac{\mathrm{d}^3 t}{E_t} \delta^4(p + q - q' - t) F(W, \Theta') \frac{\bar{q}}{\Gamma(m)} \sigma_s(m) \, \mathrm{d}m . \tag{A.5}$$

This is exactly the relativistic formulation of L. and S. hypothesis. The partial cross section for production is given by the product of two factors: a two body cross section for the production of a « particle », the « isobar », of variable mass m, multiplied by the probability of « isobar » formation.

In the limit of very narrow resonance:

$$\frac{\bar{q}\sigma_s(m)}{\Gamma(m)} \to \delta(m - m_r) .$$

As it should be from physical intuition the $\mathrm{d}\sigma_{\text{prod}}$ is reduced to the cross section for the production of a particle of mass m_r.

Appendix II

For each spectrum one has always to consider the linear combination of the same three terms (corresponding to the « extra-π » the « decay-π » and the interference), with coefficients given by the following Table:

$T = \frac{3}{2}$ (*state of production*)

	Extra-π	Decay-π	Interference
A	4/15	3/5	$-2/5$
B	3/5	4/15	$-2/5$

$$A \to \begin{cases} \pi^+ \text{ in the reaction } & \pi^+ + p \to p + \pi^+ + \pi^0 \\ \pi^- \text{ in the reaction } & \pi^- + p \to n + \pi^+ + \pi^- \end{cases}$$

$$B \to \begin{cases} \pi^0 \text{ in the reaction } & \pi^+ + p \to p + \pi^+ + \pi^0 \\ \pi^+ \text{ in the reaction } & \pi^- + p \to n + \pi^+ + \pi^- \end{cases}$$

$T=\frac{1}{2}$ (*state of production*)
(reaction: $\pi^-+p\to n+\pi^++\pi^-$)

	Extra-π	Decay-π	Interference
π^+	1/2	1/18	1/6
π^-	1/18	1/2	1/6

RIASSUNTO

In questo lavoro si propone un modello per calcolare gli effetti di interferenza tra i diversi canali isobarici nella produzione di pioni. Per energie intorno al GeV gli spettri delle particelle prodotte sono simili a quelli calcolati da Lindenbaum e Sternheimer, mentre a energie più basse si hanno differenze sostanziali.

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A Compact Expression for S-Matrix Elements in Theories with Several Interacting Fields.

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(ricevuto il 1° Aprile 1960)

Summary. — A compact method of expressing the n-th order S-matrix element in quantum electrodynamics has been given by CAIANIELLO. The generalization of this formula to n-interacting fields of arbitrary character is considered here. The specific example of the quantum electrodynamics of spin-0 mesons is treated. Although only two fields are present there are several interactions because of the e^2 term in the current and because of the meson-meson renormalization term. Using these methods a new demonstration is given of the theorem that the terms in the expansion of the S-matrix that depend on the orientation of a family of space-like surfaces vanish identically.

1. – Introduction.

An explicit form for the n-th order matrix element in quantum electrodynamics has been given by CAIANIELLO [1],

$$M_{FI} = (-1)^{p_{FI}} \frac{1}{\sqrt{\sigma_{I'}! \dots \sigma_{\beta'}! \, \tau_I! \dots \tau_\alpha!}} \sum \frac{e^N}{N!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n \sum r^{\mu_1}_{\alpha_1\beta_1} \dots r^{\mu_N}_{\alpha_N\beta_N} \cdot \\ \cdot [\xi_1 \dots \xi_n z^{(1)} \dots z^{(P_0)}] \begin{pmatrix} \xi_1 \dots \xi_N & v^{(1)} \dots v^{(N_0)} \\ \xi_1 \dots \xi_N & u^{(1)} \dots u^{(N_0)} \end{pmatrix} . \tag{1}$$

(*) Fulbright Fellow on leave from Princeton University, Princeton, N. J.

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(1) E. R. CAIANIELLO: *Nuovo Cimento*, **11**, 492 (1954), referred to as I.

There are a initial photons in α distinct states and b final photons in β distinct states. The factorial denominators give the correct normalization in the case of multiple occupation of a single state. The square and round brackets are hafnians and determinants and are expanded according to the rules given in I. The u's, v's and z's stand for the properly normalized electron and photon states as described in I. It is clear that with minor changes in the interpretation of the symbols the present formula is valid for any boson-fermion pair interacting via the usual Yukawa interaction, although the four boson vertex is not included. In the present work the above formula is generalized to the case of several interacting fields of arbitrary character.

The interaction hamiltonian is assumed to have the form

$$\mathscr{H}_I = \sum_{i=1}^{a} \lambda_i H_i \,, \tag{2}$$

where each of the fields may enter into several of the H's.

In the following we will give the formula equivalent to (1) for this case and illustrate it with the example of the electrodynamics of spin-0 mesons.

2. – The formula for the S-matrix.

The S-matrix is given by (²)

$$S = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int \mathrm{d}x_1 \dots \int \mathrm{d}x_n P(\mathscr{H}_I(x_1) \dots \mathscr{H}_I(x_n)) \,. \tag{3}$$

This must be rearranged as a multiple series in each of the coupling constants. After the definition of $\mathscr{H}_I$ is introduced we obtain

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \dots \int \mathrm{d}x_n \, P \left(\prod_{j=1}^{n} \sum_{i=1}^{a} \lambda \, H_i(x_j) \right) \,. \tag{4}$$

It should be noticed that the variables x are dummy variables of integration and may be permuted at will. This will be taken advantage of to put the time ordered product into a standard form. The H_1's of which there are r_1 will have as variables $x_1 \dots x_{r_1}$. Since there are $\binom{n}{r_1}$ ways of associating r_1 variables x with the H_1's this term must be multiplied by $\binom{n}{r_1}$. The H_2's of which there are $r_2 - r_1$ will be associated with $x_{r_1+1} \dots x_{r_2}$ and the combinatorics generates a factor $\binom{n-r_1}{r_2-r_1}$. Similarly H_b takes the variables $x_{r_{b-1}+1} \dots x_{r_b}$

(²) F. J. Dyson: *Phys. Rev.*, 5, 486 (1949).

(where $r_a = n$ and $r_0 = 1$) and there are $\binom{n - r_{b-1}}{r_b - r_{b-1}}$. Thus the expression for the S-matrix becomes

$$(5)\qquad S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{r_{a-1}=0}^{n} \sum_{r_{a-2}=0}^{r_{a-1}} \dots \sum_{r_1=0}^{r_1} \frac{n!}{r_1!\,(r_2 - r_1)!\,(r_3 - r_2)!\,\dots\,(r_{a-1} - r_{a-2})!\,(n - r_{a-1})!} \cdot$$

$$\cdot\, \lambda_1^{r_1} \lambda_2^{r_2 - r_1} \dots \lambda_a^{n - r_{a-1}}\, P\Big(\prod_{b=1}^{a} H_b(x_{r_{b-1}+1}) \dots H_b(x_{r_b})\Big)\,.$$

The use of the new variables of summation

$$s_1 = r_1\,, \qquad s_2 = r_2 - r_1\,, \qquad \dots\,, \qquad s_a = n - r_{a-1}$$

gives

$$(6)\qquad S = \sum_{s_1=0}^{\infty} \dots \sum_{s_a=0}^{\infty} \frac{(i\lambda_1)^{s_1}}{s_1!} \dots \frac{(-i\lambda_a)^{s_a}}{s_a!} \int \mathrm{d}x_1 \dots \int \mathrm{d}x_n\, P\Big(\prod_{b=1}^{a} H_b(x_{r_{b-1}+1}) \dots H_b(x_{r_b})\Big)\,,$$

where r's and n have in part been retained for convenience.

The evaluation of the time ordered product now proceeds by the methods of I. Each H consists of some product of fields with perhaps derivatives if this type of coupling is involved. Hence for the evaluation of the time ordered product it is necessary to evaluate several expressions of the form

$$P(\varphi_{k_1}(x_1) \dots \varphi_{\alpha_k}(x_k) \varphi^{+}_{\alpha_{k+1}}(x_{k+1}) \dots \varphi^{+}_{\alpha_l}(x_l)\, \partial\varphi_{\alpha_{l+1}}(x_{l+1}) \dots \partial\varphi_{\alpha_m}(x_m)\, \partial\varphi^{+}_{\alpha_{m+1}}(x_m) \dots \partial\varphi^{+}_{\alpha_n}(x_n))\,,$$

between initial and final states. Since the derivatives merely factor out and are applied to the singular function they will not be included any longer. The evaluation of such time ordered products may be easily inferred from the work in I. The most general possible result for an integral spin field is a hafnian as occurred in the electromagnetic case. If the Bose field is complex the hafnian degenerates into a permanent. This case was not encountered in I but it follows in exact analogy to the rewriting of the pfaffian as a determinant and occurs because the time ordered products must contain the field and its adjoint not just the field or just the adjoint to be $\neq 0$. For a real boson field the result given by CAIANIELLO for the electromagnetic field is applicable.

$$(7)\qquad \langle \psi_F | \varphi(x_1) \dots \varphi(x_n) | \psi_I \rangle = \frac{1}{\sqrt{\sigma_1! \dots \sigma_\beta!\, \tau_1 \dots \tau_\gamma!}}\, [x_1 \dots x_n\, z^{(1)} \dots z^{(P_0)}]$$

If the bose field is complex this becomes,

$$(8)\qquad \langle \Psi_F | P(\Phi(x_1) \dots \Phi(x_N) \Phi^{+}(x_{N+1}) \dots \Phi^{+}(x_{2N}) | \Psi_I \rangle =$$

$$= \frac{1}{\sqrt{\alpha_1!\, \alpha_p! \dots \beta_1! \dots \beta_q!\, r_1! \dots r_r!\, \delta_1! \dots \delta_s!}} \begin{bmatrix} x_1 \dots x_N & w^{(1)} \dots w^{(N_0)} \\ x_{N+1} \dots x_{2N} & u^{(1)} \dots u^{(N_0)} \end{bmatrix}\,.$$

The v's are the initial negative and complex conjugate final positive wave functions. The u's are the initial positive and complex conjugate final negative wave functions.

The α, β, γ, and δ count the numbers of particles in the initial and final states. The proof of (8) is in exact analogy to that for (7) with the added observation that because of conservation of charge the hafnian may be rewritten as a permanent.

Since Fermi fields occur with their adjoint, Caianello's formula is adequate for their matrix elements,

$$(9)\qquad \langle\psi_F|P(\psi(x_1)\dots\psi(x_N)\psi^+(x_{N+1})\dots\psi^+(x_{2N}))|\psi_I\rangle = \\ = (-1)^{\nu_F}\begin{pmatrix} x_1 & \dots\, x_N & v^{(1)}\dots v^{(N_0)} \\ x_{N+1} & \dots\, x_{2N} & u^{(1)}\dots u^{(N_0)}\end{pmatrix}.$$

The manner in which (7), (8) and (9) are substituted into (6) will depend on the details of the theory. These will be illustrated in the next section.

3. – Quantum electrodynamics of spin-0 mesons.

Detailed discussions of the perturbative expansion and renormalization of this theory have been given by ROHRLICK [3], SALAM [4], and MATTHEWS [5]. In the present treatment a compact expression will be given for the unrenormalized series. The interaction density is

$$(10)\qquad H(x) = ieA_\mu\left(\varphi\frac{\partial\varphi^*}{\partial x_\mu} - \varphi^*\frac{\partial\varphi}{\partial x_\mu}\right) + e^2[A^2 + (n_\mu A_\mu)^2]\varphi^*\varphi + \lambda\varphi^{*2}\varphi^2\,.$$

It will be most convenient to consider each monomial a separate term, *i.e.* there are five distinct interactions, two with coupling e two with e^2 and finally one with λ. According to (6) the S-matrix is given by

$$(11)\qquad S = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_3=0}^{\infty}\sum_{s_4=0}^{\infty}\sum_{s_5=0}^{\infty}\frac{e^{s_1}}{s_1!}\frac{(-e)^{s_2}}{s_2!}\frac{(-ie^2)^{s_3}}{s_3!}\frac{(-ie^2)^{s_4}}{s_4!}\frac{(-i\lambda)^{s_5}}{s_5!}\cdot \\ \cdot\int\mathrm{d}x_1\dots\int\mathrm{d}x_n\,P\left(A_{\mu_1}(x_1)\,\varphi(x_1)\frac{\partial\varphi}{\partial x_{\mu_1}}\dots\varphi^2(x_n)\,\varphi^{*2}(x_n)\right).$$

It is now necessary to apply equations (7) and (8) to express the time

(3) F. ROHRLICK: *Phys. Rev.*, **80**, 666 (1950).
(4) A. SALAM: *Phys. Rev.*, **86**, 731 (1952).
(5) P. T. MATTHEWS: *Phys. Rev.*, **80**, 292 (1950).

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ordered product. The photon field expectation value is

(12) $$\langle\psi_F|P(A_{\mu_1}(x_1)\dots A_{\mu_{s_1+s_2}}(x_{s_1+s_2})A_{\mu_{s_1+s_2+1}}(x_{s_1+s_2+1})A_{\mu_{s_1+s_2+1}}(x_{s_1+s_2+1})\dots$$
$$\dots A_{\mu_{s_1+s_2+s_3}}(x_{s_1+s_2+s_3})A_{\mu_{s_1+s_2+s_3}}(x_{s_1+s_2+s_3})(n_{\mu_{s_1+s_2+s_3+1}}A_{\mu_{s_1+s_2+s_3+1}}(x_{s_1+s_2+s_3+1}))^2\dots$$
$$\dots(n_{\mu_{s_1+s_2+s_3+s_4}}A_{\mu_{s_1+s_2+s_3+s_4}}(x_{s_1+s_2+s_3+s_4}))^2|\psi_I\rangle \doteq$$
$$=\frac{1}{\sqrt{\alpha_1!\dots\beta_1!\dots}}[1\;2\dots s_1+s_2\;\; s_1+s_2+1\;\; s_1+s_2+1\dots\;\; s_1+s_2+s_3+s_4$$
$$s_1+s_2+s_3+s_4\;Z^{(1)}\dots Z^{(P_0)}]\,.$$

There are some Kronecker δ's and n_μ's which have been suppressed that are necessary to take into account the quadratic contributions.

The contribution of the meson field is

(13) $$\langle\psi_F|P\Bigg(\Phi(x_1)\frac{\partial\Phi^*}{\partial x_{\mu_1}}\dots\varphi(x_s)\frac{\partial\varphi^*}{\partial x_{\mu_s}}\Phi^*(x_{s+1})\frac{\partial\varphi}{\partial x_{\mu_{s+1}}}\dots\Phi^*(x_{(s_1+s_2)})\frac{\partial\varphi}{\partial x_{\mu_{s_1+s_2}}}\cdot$$
$$\cdot\Phi^*(x_{s_1+s_2+1})\Phi(x_{s_1+s_2+1})\dots\Phi^*(x_{s_1+s_2+s_3+s_4})\Phi(x_{s_1+s_2+s_3+s_4})\cdot$$
$$\cdot\Phi^{*2}(x_{s_1+s_2+s_3+s_4+1})\Phi^2(x_{s_1+s_2+s_3+s_4+1})\dots\Phi^{*2}(x_{s_1+s_2+s_3+s_4+s_5})\Phi^2(x_{s_1+s_2+s_3+s_4+s_5})\Bigg)|\psi_I\rangle =$$
$$=\begin{bmatrix}1\dots s_1\;\;\partial\mu_{s_1+1}s_1+1\dots\partial\mu_{s_1+s_2}s_1+s_2\;\; s_1+s_2+1\;\dots\; s_1+s_2+s_3+s_4\\ \partial_{\mu_1}1\dots\partial\mu_{s_1}s_1\;\; s_1+1\dots\qquad s_1+s_2\;\; s_1+s_2+1\;\dots\; s_1+s_2+s_3+s_4\end{bmatrix.}$$
$$\begin{matrix}s_1+s_2+s_3+s_4+1\;\; s_1+s_2+s_3+s_4+1\;\dots\\ s_1+s_2+s_3+s_4+1\;\; s_1+s_2+s_3+s_4+1\;\dots\end{matrix}$$
$$\begin{bmatrix.}\dots\; s_1+s_2+s_3+s_4+s_5 - s_1+s_2+s_3+s_4+s_5\,w^{(1)}\dots w^{(N_0)}\\ \dots\; s_1+s_2+s_3+s_4+s_5 - s_1+s_2+s_3+s_4+s_5\,u^{(1)}\dots u^{(N_0)}\end{bmatrix}.$$

The symbols have the usual meaning except that (3)

$$[i,\,\partial\mu_j j] = \partial\mu_j\,\Delta_F(x_i - x_j)$$

and

(14) $$[\partial\mu_i i,\,\partial\mu_j j] = \partial\mu_i\,\partial\mu_j\,\Delta_F(x_i - x_j) - in_\mu n_\nu\,\delta(x_i - x_j)\,.$$

Thus except when two derivatives appear inside a time ordered product the derivatives may be moved to the outside. Next the permanent will be expanded to bring out explicitly all of the normal dependent terms. In the permanent the terms in the minor formed from rows s_1+1 to s_2 and columns 1 to s_1 are binomials. In any expansion either the first term $\partial_\mu\partial_\nu\Delta_F$ or the second $-in_\mu n_\nu\delta$ can appear. Thus the permanent has an expansion in which some number q of rows and columns contribute the δ-function and the remainder the twice differentiated propagator. Using the symmetry properties of the hafnian and the fact that the elements of the permanent are functions

of dummy variables the δ-functions part can always be permuted to consist of the q rows s_1+s_2-q+1 to s_1+s_2 and the column s_1-q+1 to s_1. There are $\binom{s_1}{q}\binom{s_2}{q}$ ways of obtaining this particular configuration and so these numerical factors must be included. Thus the permanent is equal to a sum of products of two permanents the first of which is obtained by leaving out all δ-function parts and q rows and columns, and the second permanent is composed exclusively of terms of the form $-in_\mu n_\nu \delta$. The sum is from $q=0$ to the minimum of s_1 and s_2. Further advantage may be taken of the symmetry to replace the expansion of the second permanent the δ-function one, by one term in its expansion for example the principal diagonal

$$q!\,(-1)^q n_{\mu_{s_1+s_2-q+1}} \cdots n_{\mu_{s_1+s_2}} n_{\mu_{s_1-q+1}} \cdots n_{\mu_{s_1}} \cdot \\ \cdot \delta(s_1+s_2-q+1,\ s_1-q+1) \ldots \delta(s_1+s_2,\ s_1)\,.$$

If these results are used to develop the permanent and advantage is taken of the δ-functions to carry out some of the integrations the S-matrix is altered by having q fewer factors of the types characterized by s_1 and s_2 and q more of those characterized by s_4 in addition to the numerical factor $q\,(-i)^q\binom{s_1}{q}\binom{s_2}{q}$. If the integrand in which all the implicit normal terms (those contained in $\partial_\mu\partial_\nu\Delta_F(x_i-x_j)$) have been extracted is called $M(s_1 s_2 s_3 s_4 s_5)$, S becomes,

$$S_{FI} = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_3=0}^{\infty}\sum_{s_4=0}^{\infty}\sum_{s_5=0}^{\infty}\sum_{q=0}^{\mathrm{Min}|s_1,s_2|} \frac{e^{s_1}}{s_1!}\frac{(-e)^{s_2}}{s_2!}\frac{(-ie^2)^{s_3}}{s_3!}\frac{(-ie^2)^{s_4}}{s_4!}\frac{(-i\lambda)^{s_5}}{s_5!}\cdot \\ \cdot\int \mathrm{d}x_1 \ldots \mathrm{d}x_n \binom{s_1}{q}\binom{s_2}{q}(-i)^q q!\, M(s_1-q\ \ s_2-q\ \ s_3\ \ s_4+q\ \ s_5)\,. \tag{15}$$

The variables should be changed so that $t_1=s_1-q$, $t_2=s_2-q$ and $t_4=s_4+q$ ars new variables of summation. To show that only the term $t_1=0$ contributes, the s_3 and s_5 portions are suppressed.

With these changes the relevant sums become

$$S_{FI} = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_3=0}^{\infty}\sum_{q=0}^{\mathrm{Min}|s_1,s_2|}\frac{e^{s_1}}{s_1!}\frac{(-e)^{s_2}}{s_2!}\frac{(-ie^2)^{s_4}}{s_4!}(-i)^q q!\binom{s_1}{q}\binom{s_2}{q}\cdot \\ \cdot M(s_1-q,\ s_2-q,\ s_4+q) = \\ = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_3=0}^{\infty}\sum_{q=0}^{\mathrm{Min}|s_1,s_2|}\frac{e^{s_1-q}}{(s_1-q)!}\frac{(-e)^{s_2-q}}{(s_2-q)!}\frac{(-ie^2)^{s_4+q}}{(s_4+q)!}\frac{(s_4!+q)!}{s_4!\,q!}\cdot \\ \cdot(-1)^q M(s_1-q,\ s_2-q,\ s_4+q) = \\ = \sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{t_4=0}^{\infty}\frac{e^{t_1}}{t_1!}\frac{(-e^2)^{t_2}}{t_2!}\frac{(-ie^2)^{t_4}}{t_4!}M(t_1,t_2,t_4)\sum_{q=0}^{t_4}\binom{t_4}{q}(-1)^q\,, \tag{16}$$

which vanishes unless $t_4 = s_4 + q = 0$. Thus all the derivatives may be applied to the propagators if all the terms $(n_\mu A_\mu)^2\varphi^*\varphi$ are dropped and the S-matrix has the form

$$(17)\qquad S_{FI} = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_3=0}^{\infty}\sum_{s_4=0}^{\infty}\frac{e^{s_1}}{s_1!}\frac{(-e)^{s_2}}{s_2!}\frac{(-ie^2)^{s_3}}{s_3!}\frac{(-i\lambda)^{s_4}}{s_4!}\int \mathrm{d}x_1 \ldots \int \mathrm{d}x_n \cdot$$

$$\cdot \frac{1}{\sqrt{\alpha!\,\beta!\,\gamma!\,\delta!\,\varepsilon!\,\zeta!}} \cdot$$

$$\cdot [1\,2\,\ldots\,s_1+s_2\;\; s_1+s_2+1\;\; s_1+s_2+1\,\ldots\, s_1+s_2+s_3\;\; s_1+s_2+s_3\;\; Z^{(1)}\ldots Z^{(P_0)}]\cdot$$

$$\cdot\,\partial_{\mu_1}\ldots\partial_{\mu_{s_1}}\partial^*_{\mu_{s_1+1}}\ldots\partial^*_{\mu_{s_1+s_2}}\begin{bmatrix}1\,\ldots\,s_1+s_2+s_2\;\; s_1+s_2+s_3+1 & s_1+s_2+s_3+1\,\ldots\,s_1+s_2+s_3+s_4\;\; s_1+s_2+s_3+s_4\;\; w^{(1)}\ldots w^{(N_0)}\\ 1\,\ldots\,s_1+s_2+s_3\;\; s_1+s_2+s_3+1 & s_1+s_2+s_3+1\,\ldots\,s_1+s_2+s_3+s_4\;\; s_1+s_2+s_3+s_4\;\; u^{(1)}\ldots u^{(N_0)}\end{bmatrix}.$$

The ∂'s operate on the rows and the ∂^*'s on the columns. The theorem has also been demonstrated by ROHRLICK ([3]).

RIASSUNTO (*)

Un metodo compatto per esprimere l'elemento di ordine n-simo della matrice S in elettrodinamica quantistica è stato dato da CAIANIELLO. Qui si considera la generalizzazione di questa formula ad n campi interagenti di carattere arbitrario. Viene trattato l'esempio specifico dell'elettrodinamica quantistica dei mesoni di spin 0. Per quanto siano presenti solo due campi, ci sono parecchie interazioni a causa del termine e^2 nella corrente e del termine di rinormalizzazione mesone-mesone. Usando questi metodi si dà una nuova dimostrazione del teorema secondo il quale i termini nello sviluppo della matrice S che dipendono dall'orientazione di una famiglia di superficie spazio-simili, si annullano identicamente.

(*) *Traduzione a cura della Redazione.*

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On the Hamilton Formalism in Space-Time.

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Summary. — In this paper the relativistic mechanics of the material particle in arbitrary co-ordinates is treated. We restrict ourselves on the symmetrical metric field. The Hamilton function H and the parameter s of the path in four-space are considered as the fifth pair of the canonical conjugate variables. We investigate the connection of this Hamilton formalism to the three-dimensional Lagrange formalism and we apply it to two special cases.

Introduction.

In paper (1) Infeld founds his considerations on the relativistic Lagrangian. About the Lagrangian he supposes that it is given either through the physics of some problem, or, as a natural generalization of the Lagrangian in classical mechanics. We suppose a special shape of the Hamilton function. For the choice of the Hamilton function in this shape following reasons are given: 1) in the Minkowski space our formalism is closely connected with quantum mechanics; 2) we suppose that the metric field plays an important part in all physical problems.

1. – The Hamilton function and equations of motion.

We assume that the Hamilton function H of a material particle, if one uses units which make the velocity of light equal 1, is

$$H = \sqrt{g_{\alpha\beta}p^\alpha p^\beta}\,, \tag{1}$$

(1) L. Infeld: *On Variational Principles in Relativistic Dynamics* in *Max Planck Festschrift* (Berlin, 1958).

where p^1, p^2, p^3, $p^4 = ip^0$ are the components of the momentum of the material particle and $g_{\alpha\beta}$ are the components of the fundamental symmetrical co-variant tensor which are the functions of the co-ordinates x^1, x^2, x^3, $x^4 = ix^0$. The canonical equations

$$\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha}, \qquad \dot{x}_\alpha = \frac{\partial H}{\partial p^\alpha}, \qquad (\alpha = 1, 2, 3, 4) \tag{2}$$

(where the dot means the derivation over the path parameter s), give for the components of the four-velocity

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}s} = \frac{\partial H}{\partial p^\alpha} = \frac{1}{H}\, p^\alpha . \tag{3}$$

For the «force» acting on the particle we get

$$\frac{\mathrm{d}p_\nu}{\mathrm{d}s} = -\frac{\partial H}{\partial x^\nu} = -\frac{1}{2H}\,\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\, p^\alpha p^\beta . \tag{4}$$

If we set into the foregoing equation for

$$\frac{\partial g_{\alpha\beta}}{\partial x^\nu} = g_{\lambda\beta}\Gamma^\lambda_{\alpha\nu} + g_{\lambda\alpha}\Gamma^\lambda_{\beta\nu}, \tag{5}$$

where $\Gamma^\mu_{\nu\lambda}$ are the Christoffel symbols of the second kind, we obtain

$$\dot{p}_\nu = -\frac{1}{H}\, p^\alpha p_\lambda \Gamma^\lambda_{\alpha\nu} . \tag{6}$$

For the metric field which is given by

$$g_{\alpha\beta} = \delta_{\alpha\beta} \begin{cases} = 1 & \alpha = \beta \\ = 0 & \alpha \neq \beta \end{cases} \tag{7}$$

we have $\dot{p}_\nu = 0$ and this is the law of inertia. By differentiation of (1) we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \frac{\partial H}{\partial p^\alpha}\,\frac{\mathrm{d}p^\alpha}{\mathrm{d}s} + \frac{\partial H}{\partial x^\alpha}\,\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}, \tag{8}$$

and owing to the canonical equations we get

$$\frac{\mathrm{d}H}{\mathrm{d}s} = 0 . \tag{9}$$

We can put

$$H = i\varkappa\,, \tag{10}$$

(where $i = \sqrt{-1}$) and consider $\varkappa$ as the rest mass of the particle.

2. – Jacobi equation and Lagrange function.

If we put

$$p_\nu = \frac{\partial\varphi}{\partial x^\nu}\,, \qquad H = \frac{\partial\varphi}{\partial s}\,, \tag{11}$$

where φ is the action function, we obtain from (1) the Jacobi equation:

$$\frac{\partial\varphi}{\partial s} - \sqrt{g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\,\frac{\partial\varphi}{\partial x^\beta}} = 0\,. \tag{12}$$

The equation (12) is a partial differential equation of the first order and its characteristic equations are the canonical equations (2) and the equation

$$\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = \sqrt{g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\,\frac{\partial\varphi}{\partial x^\beta}} - \frac{\partial\varphi}{\partial x^\nu}\,\frac{\partial}{\partial(\partial\varphi/\partial x^\nu)}\sqrt{g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\,\frac{\partial\varphi}{\partial x^\beta}}\,. \tag{13}$$

From (13) we see that

$$\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = 0\,. \tag{14}$$

If we define the Lagrange function $L^{(\mathrm{re})}$ as

$$L^{(\mathrm{re})} = \frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}\,, \tag{15}$$

we have owing to (14) that

$$L^{(\mathrm{re})} = 0\,. \tag{16}$$

Or we can write too

$$\int\limits_{s_0}^{s} L^{(\mathrm{re})}\,\mathrm{d}s = \varphi = \text{constant}\,. \tag{17}$$

3. – The three dimensional Lagrange function.

We can write the action function φ in the shape

$$\varphi = W + Hs\,. \tag{18}$$

Here W is a function of co-ordinates. If we put (18) in (12) we obtain

$$H^2 = g^{\alpha\beta}\frac{\partial W}{\partial x^\alpha}\frac{\partial W}{\partial x^\beta}\,. \tag{19}$$

From (18) we have:

$$\mathrm{d}\varphi = \mathrm{d}W + H\,\mathrm{d}s\,. \tag{20}$$

Owing to (14) $\mathrm{d}\varphi = 0$ and we obtain

$$\mathrm{d}W = -H\,\mathrm{d}s\,. \tag{21}$$

We can put the following expression for $\mathrm{d}W$ in the foregoing equation:

$$\mathrm{d}W = g^{\alpha\beta}\frac{\partial W}{\partial x^\alpha}\,\mathrm{d}x_\beta\,. \tag{22}$$

We transcribe (21) in:

$$g^{\alpha\beta}\frac{\partial W}{\partial x^\alpha}\,\mathrm{d}x_\beta = \varkappa\,\mathrm{d}\tau\,. \tag{23}$$

Here $i\varkappa = H$, $i\,\mathrm{d}\tau = \mathrm{d}s$. If we define the three-dimensional Lagrange function in the following manner:

$$L^{(\mathrm{cl})} = i\,\frac{\mathrm{d}\tau}{\mathrm{d}x_4}\,\varkappa = \frac{\mathrm{d}\tau}{\mathrm{d}x_0}\,\varkappa\,, \tag{24}$$

we have from (23) that

$$L^{(\mathrm{cl})} = i\left(\sum_{l,k=1}^{3} g^{kl}\frac{\partial W}{\partial x^l}\frac{\mathrm{d}x_k}{\mathrm{d}x_4} + \sum_{k=1}^{3} g^{4k}\frac{\partial W}{\partial x^4}\frac{\mathrm{d}x_k}{\mathrm{d}x_4} + \sum_{\alpha=1}^{4} g^{\alpha 4}\frac{\partial W}{\partial x^\alpha}\right). \tag{25}$$

If we put for $g^{\alpha\beta}$ from (7), the three-dimensional Lagrange function will take the following shape:

$$L^{(\mathrm{cl})} = \frac{\partial W}{\partial x^1}\frac{\mathrm{d}x_1}{\mathrm{d}x_0} + \frac{\partial W}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x_0} + \frac{\partial W}{\partial x_3}\frac{\mathrm{d}x_3}{\mathrm{d}x_0} + \frac{\partial W}{\partial x^0}\,. \tag{26}$$

Here dx_1/dx_0, dx_2/dx_0, dx_3/dx_0 are the components of the three-velocity and the partial derivations of the function W are constants, if W is the complete integral of equation (19).

4. – Special cases.

As our first special case we shall consider a particle which is under the influence of gravitation and electromagnetic field. For the momentum of this particle we take:

$$\Pi^\alpha = p^\alpha - eA^\alpha \,. \tag{27}$$

Here e is the charge of the particle and $A^\lambda(x^\nu)$ are the components of the four-potential of the electromagnetic field. The Hamilton function is given by

$$H = \sqrt{g_{\alpha\beta}\Pi^\alpha\Pi^\beta} \tag{28}$$

and from the canonical equations we obtain

$$\frac{d\Pi_\alpha}{ds} = -\frac{1}{H}\left(\Pi_\varrho \Pi^\beta \Gamma^\varrho_{\beta\alpha} - e\Pi_\beta \frac{\partial A^\beta}{\partial x^\alpha}\right), \qquad \frac{dx^\alpha}{ds} = \frac{\Pi^\alpha}{H} \,. \tag{29}$$

Introducing in the foregoing equations $ds = \pm\, i\, d\tau$ and $H = i\varkappa$ we have:

$$\frac{d\Pi_\alpha}{d\tau} = \pm\frac{1}{\varkappa}\left(\Pi_\varrho \Pi^\beta \Gamma^\varrho_{\beta\alpha} - e\Pi_\beta \frac{\partial A^\beta}{\partial x^\alpha}\right), \qquad \frac{dx^\alpha}{d\tau} = w^\alpha = \pm\frac{\Pi^\alpha}{\varkappa} \,. \tag{30}$$

Further we shall consider the equations (30) with the sign $+$ on the right hand side. Owing to (27) and

$$\frac{dA_\alpha}{d\tau} = A\alpha{:}\varrho \frac{dx^\varrho}{d\tau} = A\alpha{:}\varrho u^\varrho \,, \tag{31}$$

($:$ means absolute derivation), we obtain from (30) for the equations of motion of the particle

$$\frac{dp_\alpha}{d\tau} = \frac{p^\lambda p_\beta}{\varkappa}\Gamma^\beta_{\lambda\alpha} - \frac{e}{\varkappa}p^\lambda A_\beta \Gamma^\beta_{\lambda\alpha} - ew^\beta\left(\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha}\right). \tag{32}$$

It is impossible to assert that these equations of motion hold good. But for metric field (7) these equations are right, because they are the usual equations of motion for a charged particle in a given electromagnetic field. In the

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first special case we made the assumption about the momentum of the particle. In the second special case we shall make an assumption about the metric field.

Second special case: the Schwarzschild line element is given by

$$ds^2 = \left(1 - \frac{2\alpha}{r}\right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta \, dr^2) + \left(1 - \frac{2\alpha}{r}\right)(dx^4)^2 . \tag{33}$$

Here α is constant. Then for the Hamilton function we have:

$$H = \left[\left(1 - \frac{2\alpha}{r}\right) p_r^2 + \frac{1}{r^2}\left(p_\vartheta^2 + \frac{1}{\sin^2\vartheta} p_r^2\right) + \left(1 - \frac{2\alpha}{r}\right)^{-1} p_4^2\right]^{\frac{1}{2}} . \tag{34}$$

Here we can put $H = im$ and consider m as the rest mass of the planet. For the Jacobi equation we find:

$$\left(\frac{\partial\varphi}{\partial s}\right)^2 = \left(1 - \frac{2\alpha}{r}\right)\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial\varphi}{\partial\vartheta}\right)^2 + \frac{1}{\sin^2\vartheta}\left(\frac{\partial\varphi}{\partial r}\right)^2\right] + \left(1 - \frac{2\alpha}{r}\right)^{-1}\left(\frac{\partial\varphi}{\partial x^4}\right)^2 . \tag{35}$$

This special case can be further treated along the same lines as in classical mechanics. We obtain the well known results of the relativistic Kepler problem.

RIASSUNTO (*)

In questo scritto viene trattata la meccanica relativistica della particella materiale in coordinate arbitrarie. Ci limitiamo al campo metrico simmetrico. La funzione hamiltoniana H ed il parametro s del percorso nello spazio quadridimensionale vengono considerate come il quinto paio di variabili canoniche coniugate. Analizziamo le connessioni di questo formalismo hamiltoniano con il formalismo tridimensionale di Lagrange e lo applichiamo a due casi particolari.

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Angular Distributions.

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Summary. — The theorem for distributions in the angles θ, φ is extended to the third Euler angle, ψ, whereupon it becomes equivalent to the rotational invariance of the responsible interactions. An interpretation of half-angle terms in the general rate formula is given.

1. – Introduction.

A formally complete representation of the content of rotational invariance is given by the Wigner-Eckart theorem. Although this theorem refers directly only to matrix elements between eigenstates of the total angular momentum J and its z component M, it may nevertheless be applied to other matrix elements, by introducing transformation matrices to and from a J, M representation.

However, when one set of states is given in a J, M representation and the other is given in terms of Euler angles and rotational invariants, the content of rotational invariance may be developed very directly. The most striking result is a restriction on the dependence on the Euler angles of the matrix elements; for fixed J, there appear only a finite number of standard functions. Such restrictions have long been known [1,2], but the arguments are simplified when presented for helicity eigenstates. The discussion here will aim at generality, in presenting results involving all three Euler angles.

(1) C. N. Yang: *Phys. Rev.*, **74**, 764 (1948).
(2) P. Morrison, in Segre's: *Experimental Nuclear Physics*, vol. 2 (New York, 1953), part VI, Sect. 1*B*.

2. – Euler angles.

Euler angles are familiar from the discussion of a rigid body in classical mechanics. One imagines the body placed in some standard or fiduciary orientation with respect to a right-handed Cartesian reference frame. The body is rotated first through angle ψ about the z axis, by which is meant a counterclockwise or positive rotation in the xy plane, and which is formally equivalent to a corresponding negative rotation of the reference frame; then through angle θ about the y axis; and finally through angle φ about the z axis. The entire operation will be designated $R(\varphi\theta\psi)$. The same operation can be applied to a fiduciary quantum-mechanical state, to generate a rotationally invariant family of states as the Euler angles φ, θ, ψ are allowed to vary.

If the original state be decomposed into its irreducible parts according to the covering group of the rotation group, *i.e.*, into eigenstates of total angular momentum J, it is seen that a rotation of 2π about any axis restores the components of integral J, and restores those of half-odd-integral J except for reversal of sign. Therefore, as φ, θ, ψ vary over a doubled Euler-angle domain, the state in all cases varies over all its rotational images. The usual convention for Euler-angle domain is $0 \leqslant \varphi < 2\pi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \psi < 2\pi$, and an appropriate doubled range may be obtained by, *e.g.*, allowing either φ or ψ the range $[0, 4\pi)$.

More particularly, consider a state of n particles of definite spin, the i-th particle with momentum $\boldsymbol{p}_i$, and with helicity (spin components along the direction of $\boldsymbol{p}_i$) λ_i, with total momentum $\sum_{i=1}^{n} \boldsymbol{p}_i = 0$. Such a state may be specified by a list of rotational invariants: the λ_i, the $\boldsymbol{p}_i \cdot \boldsymbol{p}_j$ — the latter being of course highly redundant for large n, and any specifications of the types of particles involved, augmented by three Euler angles. The latter may be assigned as follows: ψ to specify the orientation of the figure of $n-1$ of the momenta about a distinguished one, and θ, φ to specify the direction of the distinguished momentum. Such assignments are particularly convenient if one wishes to describe the angular distribution of one particular final-state particle, although other assignments could be equally interesting if one is interested in the entire figure of n momenta.

Exact phase conventions are easily fixed ([3]). Pick creation operators for particles at rest, spin-quantized on the z axis, with Condon-Shortley conventions for the infinitesimal rotation operators J_x, J_y. Then apply an active Lorentz transformation to produce a creation operator for a particle of momentum p along the z axis, and finally rotate with the operator $R(\varphi_p, \theta_p, 0)$

([3]) M. JACOB and G. C. WICK: *Ann. Phys.*, **7**, 404 (1959), hereafter designated JW.

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to produce the creation operator appropriate to an arbitrary direction. $R(\varphi_p, \theta_p, -\varphi_p)$ is an alternate choice used by JW, equally applicable here; any definite convention for ψ_p will do. Our fiducial state is produced by n successive creation operators of this form acting on a vacuum state, and the general state is obtained by applying $R(\varphi\theta\psi)$ to the entire fiducial state.

Thus we have produced a rather general example of how co-ordinates may be so specified as to yield a system of states labelled exclusively by a list β of rotational invariants or «scalars» and Euler angles, φ, θ, ψ. Since the scalars don't enter actively in our arguments, no detailed discussion of their redundancy in particular examples will be presented, but redundancy involving the Euler angles will claim some of our attention.

For $n \geqslant 3$ particles, we have, except in very special cases which we will not discuss, a configuration which requires all three angles for the description of its orientation. Thus, θ, φ may be used as spherical angles for the momentum of largest absolute value, and ψ as the angle from the plane of longitude through that momentum to the plane determined by the two largest momenta; ψ is then the angle east of south of the second momentum's projection on a sphere at the point θ, φ determined by the first. There is, then, no linear dependence among the states given by the φ, θ, ψ of the Euler-angle domain, but the states given by φ, θ, ψ in the other half of a doubled domain are either $\pm$ the corresponding states, obtained from them by a 2π rotation, according as the number of fermions is even or odd. This redundancy is not necessarily present for states where the number of fermions is not sharp.

For a two-body state of particles of definite spin and helicity, there is much more redundancy, so that in this discussion, it appears as a degenerate case. In fact, the ψ rotation leads only to a phase factor. Since the absolute value of the momentum is not in this case available for distinction of particles, there is even further redundancy for the case of two identical particles of the same helicity, for which a complete but independent set of states is obtained by cutting the Euler-angle domain in half.

3. – The general theorem.

The matrix element of a rotationally invariant operator S between a state $|JM\alpha\rangle$ specified by the total angular momentum J, its z component M, and a list α of rotational invariants, and a state $|\varphi\theta\psi\beta\rangle$ specified by Euler angles and a list β of rotational invariants, is always of form

$$(1)\qquad \left\{\begin{aligned} \langle\varphi\theta\psi\beta|S|JM\alpha\rangle &= \sum_{M'} (\beta|JM'\alpha)\exp[iM'\psi]\, d^J_{M'M}(-\theta)\exp[iM\varphi] \\ &= \sum_{M'} (-)^{M-M'}(\beta|JM'\alpha)\, D^J_{-M,-M'}(\varphi\theta\psi)\,, \end{aligned}\right.$$

where the $(\beta|JM\alpha)$ are the values of the matrix elements for all Euler angles zero, and the d and D functions are the matrices for active rotations defined in JW; namely,

$$R(\varphi\theta\psi) = \exp[-iJ_z\varphi]\exp[-iJ_y\theta]\exp[-iJ_z\psi]\,, \tag{2}$$

$$R(\varphi\theta\psi)|JM\rangle = \sum_{M'} D^J_{M'M}(\varphi\theta\psi)|JM'\rangle\,, \tag{3}$$

$$\exp[-iJ_y\theta]|JM\rangle = \sum_{M'} d^J_{M'M}(\theta)|JM'\rangle\,, \tag{4}$$

so that

$$D^J_{M'M}(\varphi\theta\psi) = \exp[-iM'\varphi]\,d^J_{M'M}(\theta)\exp[-iM\psi]\,. \tag{5}$$

The result (1) is obtained as follows. $|\varphi\theta\psi\beta\rangle = R(\varphi\theta\psi)|000\beta\rangle$, so that $\langle\varphi\theta\psi\beta| = \langle 000\beta|R^{-1}(\varphi\theta\psi)$. The rotation operator thus extracted is commuted with the rotationally invariant operator S, and then in the form

$$R^{-1}(\varphi\theta\psi) = \exp[iJ_z\psi]\exp[iJ_y\theta]\exp[iJ_z\varphi]$$

is applied via eq. (4) to $|JM\alpha\rangle$, to obtain (1). The second form is obtained by means of the identity

$$d^J_{M'M}(\theta) = (-)^{2J+M+M'}\,d^J_{M'M}(-\theta) \tag{6}$$

supplemented by the identity

$$d^J_{M'M}(\theta) = (-)^{M'-M}\,d^J_{MM'}(\theta) \tag{7}$$

of JW, eq. (A.1).

The importance of the result lies in the fact that only a small number of functions of the Euler angles is involved, for fixed J. Such qualitative remarks will also be seen to hold for appropriate rates.

4. – Relations among the generators.

It is almost obvious that (1) not only follows from rotational invariance of S, but also implies rotational invariance. More precisely, we inquire whether the generators $(\beta|JM\alpha)$ of the matrix element according to (1) may be chosen as independent complex numbers, or what relations they must satisfy, to assure rotational invariance and a consistent definition of the matrix elements.

Suppose, first, that the states $|\varphi\theta\psi\beta\rangle$ are linearly independent, so that (1)

may be used as a definition of its left-hand side for arbitrary complex numbers $(\beta|JM\alpha)$ on the right. Then it is easy to prove that the matrix so defined is rotationally invariant. Explicitly we ask

$$(8) \qquad \langle\varphi_1\theta_1\psi_1\beta|R^{-1}(\varphi_2\theta_2\psi_2)S|JM\alpha\rangle = \langle\varphi_1\theta_1\psi_1\beta|S\sum_{M'} D^{-1}_{M'M}{}^J(\varphi_2\theta_2\psi_2)|JM'\alpha\rangle ?$$

The l.h.s. is equal $\langle\Phi\Theta\Psi\beta|S|JM\alpha\rangle$, where Φ, Θ, Ψ are the angles achieved by the successive operation $R(\varphi_2\theta_2\psi_2)R(\varphi_1\theta_1\psi_1) = R(\Phi\Theta\Psi)$. By (1), it follows that the l.h.s. is equal to

$$\langle 000\beta|S\sum_{M'} D^{-1}_{M'M}{}^J(\Phi\Theta\Psi)|JM'\alpha\rangle = \langle 000\beta|S\sum_{M'M''} D^{-1}_{M'M''}{}^J(\varphi_1\theta_1\psi_1)D^{-1}_{M''M}{}^J(\varphi_2\theta_2\psi_2)|JM'\alpha\rangle .$$

But this is also the result obtained by applying (1) directly to the r.h.s. of (8).

We will now examine cases where the $|\varphi\theta\psi\beta\rangle$ are not linearly independent. We will always imagine redundant values of β to be cast out, so that we address ourselves only to redundancy involving the Euler-angle variables. Of course, both sides of (1) are periodic with any doubled Euler-angle domain for period, so that the general restriction of (1) as a definition to a single doubled Euler-angle domain involves no restriction on the generators.

The first material restrictions we discuss arise in cases of states of n particles of definite spin, where the states for $\varphi\theta\psi$ in a doubled Euler-angle domain depend on those in an ordinary Euler-angle domain. Eq. (1) may still be used as a definition of its l.h.s. if restricted to an ordinary Euler-angle domain, provided that there are not further redundancies. If the number of fermions is even, then the l.h.s. reproduces itself in the other half of a doubled Euler-angle domain, but the r.h.s. does not, unless it contains no term of half-odd-integral J. If the number of fermions is odd, the l.h.s. reproduces itself with a sign reversal, but the r.h.s. does not, unless it contains no term of integral J. The restrictions on the generators $(\beta|JM\alpha)$ are, therefore, that all those with half-odd-integral or integral J vanish, respectively. If the $|\varphi\theta\psi\beta\rangle$ restricted to an ordinary Euler-angle domain are linearly independent, there are no further relations implied by rotational invariance.

For the case of $n=2$ particles with definite spins and helicities, we noted in Section 2 that there is further redundancy, namely, that ψ is a superfluous co-ordinate. If there is no redundancy beyond this, (1) can be taken as a definition of its l.h.s. for $\psi=0$. For rotational invariance, it must however hold for all ψ. If the fiducial state has particles of helicities λ_1, λ_2 and absolute value of momentum p moving respectively in the $\pm z$ directions, we write the fiducial state as $|000p\lambda_1\lambda_2\rangle$. The ψ dependence of the l.h.s. is, then,

$$\langle\varphi\theta\psi p\lambda_1\lambda_2|S|JM\alpha\rangle = \exp[i(\lambda_1-\lambda_2)\psi]\langle\varphi\theta 0 p\lambda_1\lambda_2|S|JM\alpha\rangle .$$

The r.h.s. of (1) is a superposition of several terms with ψ dependence $\exp[iM'\psi]$, so that (1) remains true for all $\psi \neq 0$ if and only if

$$(p\lambda_1\lambda_2|JM'\alpha) = 0 \qquad \text{when } M' \neq \lambda_1 - \lambda_2, \tag{9}$$

a relation which has the obvious meaning of conservation of the z component of angular momentum in transitions to the fiducial state.

There is still further redundancy for $n = 2$ identical particles of the same helicity $\lambda_1 = \lambda_2 = \lambda$. In the Appendix it is shown that

$$|0\pi 0 p\lambda\lambda\rangle = |000p\lambda\lambda\rangle\,. \tag{10}$$

All the further redundancy is obtained by applying a rotation to both sides of (10). Since (1) is rotationally invariant, it is necessary to satisfy the identification implied by (10) only for one example of identified states. All further relations among the generators must therefore follow from the requirement

$$\langle 0\pi 0 p\lambda\lambda|S|J0\alpha\rangle = \langle 000p\lambda\lambda|S|J0\alpha\rangle \equiv (p\lambda\lambda|J0\alpha)\,. \tag{11}$$

The l.h.s. of (11) is, by (1), equal to $(p\lambda\lambda|J0\alpha)d^J_{00}(-\pi)$. But $d^J_{00}(-\pi) = (-)^J$, which with (11) shows that the relations are precisely that

$$(p\lambda\lambda|J0\alpha) = 0 \quad \text{unless } J \text{ is even}, \tag{12}$$

a result which appears in JW after eq. (47).

If S is to commute with operators P other than rotations, there will be further restrictions among the generators. The additional relations all follow from

$$\langle 000\beta|P^{-1}SP|JM\alpha\rangle = (\beta|JM\alpha)\,, \tag{13}$$

provided $P^{-1}SP$ is rotationally invariant, which will be true if P commutes with rotations. The case of parity is such an example, but these relations usually involve the scalars β in such a way that they may be satisfied by a restriction of the list β of scalars. In general, it is only for $n \leqslant 3$ particles and all helicities zero that the parity image is equal to a rotate of the original state, up to a phase factor. These relations will not be worked out; for the case of $n = 2$, see JW.

5. – Rates.

The matrix elements of eq. (1) will be introduced into the general formula,

$$R_{FI}(\varphi\theta\psi) = \sum_{\alpha\beta\alpha'\beta' JKm_1m_2} \langle\varphi\theta\psi\beta|S|Jm_1\alpha\rangle \cdot \cdot\langle\varphi\theta\psi\beta'|S|K, -m_2, \alpha'\rangle^* \varrho^I_{J,m_1,\alpha;K,-m_2,\alpha'}\,\varrho^F_{\beta',\beta}\,, \tag{14}$$

for a rate per $2\pi\times$volume of phase space, or an absolute rate if conservation δ functions are kept in the matrix elements, from an initial mixture given by the density matrix ϱ^I to a final mixture with sharp values for the Euler angles $\varphi\theta\psi$, but which may otherwise be a mixture, so that the final mixture is given by the operator $|\varphi\theta\psi\beta'\rangle\varrho^F_{\beta',\beta}\langle\varphi\theta\psi\beta|$. The use of an initial mixture is too well known to deserve comment; the use of a final mixture corresponds to an experiment where several linearly independent system states may correlate with the same indication of a measuring apparatus.

From (1),

$$R_{FI}(\varphi\theta\psi)=\sum_{\alpha\beta\alpha'\beta'JKm_1m_2m_1'm_2'}\varrho^I_{J,m_1,\alpha;K,-m_2,\alpha'}\,\varrho^F_{\beta',\beta}(\beta|Jm_1'\alpha)(\beta'|K,-m_2',\alpha')^*\cdot$$
$$\cdot\exp[i(m_1'+m_2')\psi]\,d^J_{m_1',m_1}(-\theta)\,d^K_{-m_2',-m_2}(-\theta)\exp[i(m_1+m_2)\varphi]\,.$$

By means of eq. (32) of JW, the product of d functions may be written as a sum of d functions,

$$d^J_{m_1',m_1}(-\theta)\,d^K_{-m_2',-m_2}(-\theta)=\sum_L c^{J\;K\;\;L}_{m_1'm_2'm_1'+m_2'}\,c^{J\;K\;\;L}_{m_1m_2m_1+m_2}(-)^{m_2-m_2'}d^L_{m_1'+m_2',m_1+m_2}(-\theta)\,,$$

where c's are Clebsch-Gordan coefficients in obvious notation. By using (6) and eq. (A.1) of JW,

$$d^L_{-m',\,m}(\theta)=d^L_{m,m'}(\theta)\,, \tag{15}$$

one obtains

$$R_{FI}(\varphi\theta\psi)=\sum_{\alpha\beta\alpha'\beta'JKLm_1m_2mm_1'm_2'm'}\varrho^I_{J,m_1,\alpha;K,-m_2,\alpha'}\,\varrho^F_{\beta',\beta}(\beta|Jm_1'\alpha)\cdot$$
$$\cdot(\beta'|K,-m_2',\alpha')^*(-)^{m_1+m_1'}c^{J\;K\;\;L}_{m_1'm_2'-m'}\,c^{J\;K\;\;L}_{m_1m_2-m}\exp[-im\varphi]\,d^L_{m,m'}(\theta)\exp[-im'\psi]\,, \tag{16}$$

which has been brought to the standard form of an expansion in the $D^L_{m,m'}(\varphi\theta\psi)$ as may be seen by comparison with (5). This formula is useful when $\varrho^I_{J\,m_1,\alpha;K,-m_2,\alpha'}(\beta|Jm_1'\alpha)(\beta'|K,-m_2',\alpha')^*$ is large only for a few J, K, as then eq. (16) involves only a small number of standard functions of the Euler angles, although in general, (16) is only a formal expansion in the complete set of functions $D^L_{m,m'}(\varphi\theta\psi)$. In fact, the Clebsch-Gordan coefficients vanish, unless

$$|J-K|\leqslant L\leqslant J+K\quad\text{and}\quad m_1+m_2+m=0\,,\quad m_1'+m_2'+m'=0\,.$$

If the initial mixture has a definite value for J, then $L\leqslant 2J$. If also all $m_1=-m_2=M$, then $m=m'=0$, so that the basic functions reduce to Legendre polynomials of degree up to $2J$, as $d^L_{00}(\theta)=P_L(\theta)$. A form with $m=m'=0$ is also obviously obtained if one is not interested in the φ and ψ distributions, and therefore integrates the rate over these angles. If one

is interested in the detailed angular distribution of one particle, one may choose its momentum to define θ, φ, and integrate over ψ, thereby obtaining $m' = 0$ and an expansion in spherical harmonics, since the $d^L_{m,0}(\theta)$ are proportional to associated Legendre polynomials, $d^L_{m,0}(\theta) = [4\pi/(2L+1)]^{\frac{1}{2}} \mathscr{P}_{L,m}(\theta)$ [4]. If one wishes to sum over final state helicities or over initial state helicities subject to definite value of total angular momentum and z component thereof, one may indicate this in the density matrices $\underset{\sim}{\varrho}'$, $\underset{\sim}{\varrho}''$, for such helicities are included in the lists α, β of scalars, or one may sum several particularized rates (16); either way, it is clear that the form of (16) remains unchanged by such summations. Other scalars, as mutual angles between momenta, and absolute values of momenta, may also be summed over, to obtain, *e.g.*, a formula of form (16) for the angular distribution of one particle in a final state, irrespective of other final-state variables.

Inequalities among the coefficients consequent on positiveness of the density matrices will not be explicitly displayed. For a partial discussion, see references [5,6].

In order to obtain a rate for a definite polarization or definite polarizations other than definite helicities, the helicity states must be appropriately superposed. If, *e.g.*, the final-state helicity quantum number λ is involved in this way, then we may use λ as one of the scalars in the list β, λ' in β', and incorporate $l^*_{\lambda'} l_\lambda$ in $\underset{\sim}{\varrho}^F_{\beta',\beta'}$, where l_λ is the coefficient of the λ helicity state in the final polarization eigenstate. If the l_λ are constants, a definite relation of polarization to orientation is implied, which is nevertheless more general than that given by pure helicity eigenstates, whereas the Euler-angle dependence of (16) is unaltered. Thus, if the l_λ device with constant l_λ is used to give a transverse polarization in the $\varphi = 0$ or x direction to a particle moving in the $+z$ direction in the fiducial state $\varphi = \theta = \psi = 0$, then the general state with $\psi = 0$ has this particle with a transverse polarization pointing due south, and for $\psi \neq 0$ the polarization is still transverse, but is oriented ψ east of due south. If the polarization is required to have other dependence on the Euler angles $\varphi\theta\psi$, then the l_λ will be functions of the Euler angles, so that the Euler-angle dependence of (16) will no longer be that explicitly displayed in (16). For an analogous explicit expression, the l_λ and $l^*_{\lambda'}$ would have to be analyzed into D functions, which would when multiplied by each other and by the explicit D functions in (16) give rise again to a sum of D functions.

Finally, there is the very important case of the 2-particle initial state, with sharp momenta along the z-axis, and sharp helicities λ_1, λ_2, the first associated with the particle moving in the $+z$ direction. In the notation and

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(4) JW, p. 426.

(5) T. D. Lee and C. N. Yang: *Phys. Rev.*, **109**, 1755 (1958).

(6) M. Tausner: Columbia University doctoral dissertation.

normalization of JW, who suppress the index p, JW (20), (22), (24), this state is

$$|00\lambda_1\lambda_2\rangle = \sum_J [(2J+1)/4\pi]^{\frac{1}{2}} |JM\lambda_1\lambda_2\rangle\,, \qquad M = \lambda_1 - \lambda_2,$$

which corresponds to the use of $(4\pi)^{-1}(2J+1)^{\frac{1}{2}}(2K+1)^{\frac{1}{2}}$ for ϱ^I and $m_1 = -m_2 = \lambda_1 - \lambda_2$, $m = 0$, in eq. (16).

6. – Interpretation of Half-Odd-Integral D Functions in the Rate.

The formal expression (16) admits non-zero terms with half-odd-integral L, when ϱ^I terms with one of J, K integral and the other half-odd-integral occur, provided that also ϱ^F include such states β that $(\beta|Jm_1'\alpha)$ of both integral and half-odd-integral J don't vanish, or more precisely, that $S^\dagger|\beta')\varrho^F_{\beta'\cdot\beta}(\beta|S$ and ϱ^I have elements between common integral and half-odd-integral J states. If we confine ourselves to states formed by creation of fermions and bosons into vacuum, this requirement means that, when (16) describes a transition probability between pure states, both initial and final states must be superpositions of states with even and odd numbers of fermions, with a definite relative phase γ between such components of a state. Since a density matrix is used to represent the correlations of a limited system with external systems or «measuring devices» in cases where the relevant matrix elements involve only limited-system variables, the use of density matrices of the type necessary to yield half-odd-integral L will arise only if there exist interactions involving such phases γ. Such interactions are not known; the content of the half-odd-integral L terms of (16) is that their existence would be equivalent to an independent physical meaning for all the co-ordinates in a doubled Euler-angle domain. The remainder of this section is devoted to the elaboration of a rather fanciful example, to make this point clear.

Let $a^\dagger$ be a creation operator for a spinless particle at rest, and let $b^\dagger$ be a creation operator for a spin $\frac{1}{2}$ particle at rest, with z component of spin $\frac{1}{2}$. Then

$$\begin{aligned}(17)\qquad & R(\alpha\beta\gamma)2^{-\frac{1}{2}}(a^\dagger + b^\dagger)\big(R(\alpha\beta\gamma)\big)^{-1} = \\ &= 2^{-\frac{1}{2}}\Big(a^\dagger + \sum_M \exp[-i\alpha M]\, d^{\frac{1}{2}}_{M,\frac{1}{2}}(\beta)\exp[-i\gamma/2]\, b^\dagger_M\Big) = \\ &= 2^{-\frac{1}{2}}\Big(a^\dagger + \sum_M D^{\frac{1}{2}}_{M,\frac{1}{2}}(\alpha\beta\gamma) b^\dagger_M\Big) \equiv q^\dagger(\alpha\beta\gamma)\,,\end{aligned}$$

where $b^\dagger_M$ is the creation operator for the fermion at rest with z component of spin M, may be regarded as the creation operator for a particle of mixed spin at rest, with its polarization pointing in the direction β, α and with γ specifying the relative phase between the fermion and boson components. For the corresponding mixed particles to appear as physical entities with well-

defined internal phase γ, it is necessary that the masses of boson and fermion components be sufficiently close that the mass difference imply not too rapid a change of γ, and that an interaction involving mixed creation and annihilation operators, $q^\dagger$ and q, occur. Thus, the decay of a heavy mixed particle into two spinless bosons and a light mixed particle could be described by formula (16), with $\varphi\theta\psi$ having the usual interpretation as Euler angles for a 3-body final state. Let θ, φ be the polar angles for the final mixed particle, let ψ specify the orientation of the decay plane, and let us assume detectors sensitive to the helicity of the fermion component and the internal phase of the mixed final particle; at each angle θ, φ, for definite fermion helicity, there is still a 2-dimensional space of mixed-particle states, and we assume a detector sensitive to a particular one of these dimensions. Such a detector could distinguish between a state and its 2π-rotate, because the change of internal phase renders the 2π-rotate orthogonal to the unrotated state. Therefore the result (16) for the rate, involving half angles in this case, is perfectly reasonable.

A detector sensitive to internal phase is also necessary to prepare the initial state. Such a detector could be imagined to be fashioned from mixed particles itself, provided that there be a mutual interaction between mixed particles depending on the difference of internal phase of the interacting particles, so that internal phases could be calibrated relative those of the mixed particles in the detector. A simpler «detection» could be imagined if the detection interactions are allowed to violate rotational invariance, although the transition operator S is not; *e.g.*, the lifetimes of mixed particles could be taken dependent on internal phase.

* * *

The general theorem for the distribution in polar angles θ, φ of one particle from the decay of an eigenstate of J and M, quoted to me by Prof. T. D. LEE, proved very valuable in simplifying model calculations of angular distributions of neutrons from μ-capture. Conversations with Dr. J. V. LEPORE on the angular distribution of a pion from antiproton annihilation were also stimulating.

APPENDIX

This is a verification that

$$(10) \qquad |0\pi 0 p\lambda\lambda\rangle = |000p\lambda\lambda\rangle ,$$

in the case of two identical particles, or see JW, p. 419.

$$(18) \qquad |000p\lambda\lambda\rangle = L_z(p)a_\lambda^\dagger L_z(-p)L_z(-p)a_{-\lambda}^\dagger L_z(p)\,|vac\rangle ,$$

where $a_\lambda^\dagger$ creates one particle of momentum 0, z component of spin λ, total spin s, and $L_z(p)$ is a Lorentz transformation along the z axis leading to a $+z$

component p of momentum for a single particle originally at rest.

$$|0\pi 0 p\lambda\lambda\rangle = R(0\pi 0)|000p\lambda\lambda\rangle\,, \tag{19}$$

and we abbreviate $R(0\pi 0) = R_y(\pi)$. Clearly,

$$R_y(\pi)L_z(p)a^\dagger_\lambda|vac\rangle = aL_z(-p)a^\dagger_{-\lambda}|vac\rangle\,, \tag{20}$$

$$R_y(\pi)L_z(-p)a^\dagger_{-\lambda}|vac\rangle = bL_z(p)a^\dagger_\lambda|vac\rangle\,, \tag{21}$$

where a and b are phase factors. By applying $R_y(\pi)$ to both sides of (20) and using (21), we find

$$R_y(2\pi)L_z(p)a^\dagger_\lambda|vac\rangle = abL_z(p)a^\dagger_\lambda|vac\rangle\,,$$

whereupon the fact that $R_y(2\pi)$ acts as ± 1 on a state of even or odd total angular momentum, respectively, gives us

$$ab = (-)^{2s}\,. \tag{22}$$

From (20), (21) and standard q-number theory conventions,

$$R_y(\pi)L_z(p)a^\dagger_\lambda L_z(-p)R_y(-\pi) = aL_z(-p)a^\dagger_{-\lambda}L_z(p)\,, \tag{23}$$

$$R_y(\pi)L_z(-p)a^\dagger_{-\lambda}L_z(p)R_y(-\pi) = bL_z(p)a^\dagger_\lambda L_z(-p)\,. \tag{24}$$

Explicitly, (19), (18) yield

$$|0\pi 0p\lambda\lambda\rangle = R_y(\pi)L_z(p)a^\dagger_\lambda L_z(-p)R_y(-\pi)R_y(\pi)L_z(-p)a^\dagger_{-\lambda}\cdot \cdot L_z(p)R_y(-\pi)|vac\rangle\,, \tag{25}$$

so that by (23), (24),

$$|0\pi 0p\lambda\lambda\rangle = ab[L_z(-p)a^\dagger_{-\lambda}L_z(p)][L_z(p)a^\dagger_\lambda L_z(-p)]|vac\rangle\,. \tag{26}$$

If the operators in brackets could be interchanged, we see by (18) that $ab|000p\lambda\lambda\rangle$ would be obtained. According to the spin-statistics relation, this can be done if we introduce a factor $(-)^{2s}$. By (22), this cancels ab, yielding (10).

RIASSUNTO (*)

Si estende il teorema per le distribuzioni secondo gli angoli θ, φ al terzo angolo di Eulero, ψ, per cui esso diviene equivalente alla invarianza rotazionale delle interazioni responsabili. Si dà una interpretazione dei termini semi-angolari nella formula del tasso generale.

(*) *Traduzione a cura della Redazione.*

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Statistical Emission of Nucleons in Reactions Produced by 14 MeV Neutrons.

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Summary. — We have analysed a certain number of nuclear reactions for medium and light nuclei ($10 < Z < 40$), particularly (n, p) reactions with 14 MeV neutrons. The expression $n(\varepsilon)/\varepsilon\sigma^*$, which is proportional to the nuclear level density, is represented fairly well, for ε in the range $(4\div 10)$ MeV, by an exponential law of the kind $e^{-\varepsilon/\theta}$, being $\theta=(1\div 1.2)$ MeV for medium nuclei ($20 < Z < 40$) and $\theta=1.45$ for lighter nuclei. σ^* has been calculated assuming $r_0=1.4\cdot 10^{-13}$ cm for neutrons and $r_0=1.6\cdot 10^{-13}$ cm for protons. On such a basis the ratio $\sigma(\mathrm{n, p})/\sigma(\mathrm{n, n_I})$ is in good agreement with the evaporative model when one considers the Q values of the reaction and the fact that the residual nucleus is even-even or odd-odd. From such ratio, calculated for even A nuclei, it is possible to obtain values of the pairing energy Δ in good agreement with that calculated by Cameron. Also for (n, 2n), (n, np) and (p, n) reactions on medium nuclei, the evaporative model gives satisfactory results for the interpretation of the cross-section variations.

1. – Introduction.

In a recent work (1), an accurate analysis was conducted on a certain number of nuclear reactions at medium energies. In particular, comparison was made among spectra of protons, neutrons and α particles when divided by the pe-

(1) L. Colli, U. Facchini, I. Iori, M. G. Marcazzan and A. M. Sona: *Nuovo Cimento*, **13**, 730 (1959).

netrability expression $\varepsilon\sigma^*$, ε being the energy of emitted particles and σ^* the so-called inverse cross-section, according to the statistical model by Weisskopf and Ewing [2]. According to this model, when the energy distribution function $n(\varepsilon)$ is divided by the expression $\varepsilon\sigma^*$, we obtain a new function $\omega(E)$, proportional to the level density of the residual nucleus at energy E, given by $E=\varepsilon_{max}-\varepsilon$, ε_{max} being the maximum emission energy.

The conclusions obtained in (1) can be summed up as follows:

A) A group of results concerning the medium nuclei ($20<Z<40$) and, in the case of particular incident energies, namely n, p reactions with 14 MeV neutrons [1,3-6], p, p′ reactions with 11.3 and 13.4 MeV protons [7], α, α′ reactions with (12 ÷ 20) MeV particles [8], n, n′ [9] and p, n [10] reactions at 14 and 15 MeV, seem to conform well to the statistical model (see Fig. 6 and 11 of [1]). From these spectra, to which corresponds a nearly-isotropic angular distribution one obtains, in fact, a series of $n(\varepsilon)/\sigma^*\cdot\varepsilon$ curves which, for values of the excitation energy E from 3 to 10 MeV, are about the same for various reactions and can therefore be presumed to represent the shape of the nuclear levels density $\omega(E)$. The values of σ^* used in [1] are the same assumed by various authors, that is, for the neutron spectrum, $\sigma^*=$ constant; for protons the values of FESHBACH, SHAPIRO and WEISSKOPF [11] listed in the volume BLATT and WEISSKOPF [12] for $r_0=1.5\cdot10^{-13}$ cm and for α particles the values used by FULLBRIGHT, LASSEN and ROY PAULSEN [8].

The shape of the curve $\omega(E)$ proves to be well represented by an exponential of the type $\exp[E/\theta]$, where θ is of the order of (1 ÷ 1.2) MeV. The hypothesis that these curves represent the densities of the nuclear levels is substantiated by recent results of ERICSON [13] who, by examining the levels of a certain number of medium nuclei, has obtained for the level densities an exponential shape of the above mentioned type and with the same values of θ.

(2) V. F. WEISSKOPF and D. H. EWING: *Phys. Rev.*, **57**, 472 (1940).

(3) D. L. ALLAN: *Nucl. Phys.*, **6**, 464 (1958); **10**, 348 (1959).

(4) P. V. MARCH and W. T. MORTON: *Phil. Mag.*, **3**, 143 (1957); **3**, 577 (1958).

(5) I. KUMABE and R. W. FINK: *Nucl. Phys.*, **15**, 316 (1960).

(6) A. H. AMSTRONG and L. ROSEN: to be published.

(7) B. L. COHEN and A. G. RUBIN: *Phys. Rev.*, **113**, 579 (1959).

(8) H. W. FULLBRIGHT, N. O. LASSEN and N. O. ROY POULSEN: *Dan. Viden.*, **31**, no. 10 (1959); N. O. LASSEN and N. O. ROY PAULSEN: *Intern. Conf. on Low Energy Nuclear Physics in Paris* (July 1958).

(9) J. BENVENISTE: *Int. Conf. of Geneva*, P/2494 (1958).

(10) P. C. GUGELOT: *Phys. Rev.*, **81**, 51 (1951).

(11) H. FESHBACH, M. M. SHAPIRO and V. F. WEISSKOPF: NYO-3077 (1954).

(12) J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952), p. 352.

(13) T. ERICSON: *Nucl. Phys.*, **11**, 481 (1959).

B) For incident energies greater than those indicated above, that is p, p′ reactions of 15 to 30 MeV ([14,15]), α, α′ reactions of 40 MeV ([16]) etc., a similar analysis was conducted in ([1]).

Of the spectra emitted at backward angles, those parts were treated in which only one particle proved to have been emitted, and which correspond to excitation energies E in the same interval from 3 to 10 MeV. The curves $n(\varepsilon)/\sigma^* \cdot \varepsilon$, were therefore obtained and compared with those $\omega(E)$ discussed in *A*).

Here, too, it is observed that these curves $n(\varepsilon)/\sigma^* \varepsilon$ expressed as a function of E increase almost exponentially with E, but the slope of these curves is inferior to those discussed previously, and is as smaller as the incidental energy is greater.

This behaviour is attributed to the increasing importance of the direct effects ([1 7]) which become dominant with respect to the statistical emission, at least as far as the considered part of the spectrum is concerned.

We may recall that this deviation from the statistical emission is accompanied, according to each different case, by an angular distribution which is more strongly forward peaked.

We may also recall that the parts of the spectrum considered in this comparison correspond to energies of the outgoing particles that increase with the incident energy, and, hence, to exit channels nearest to the entrance channel.

All these considerations justify the explanation of these effects as being due to direct effects, perhaps emissions of the instantaneous type as brought into evidence in recent experiments ([17,18]) or, perhaps ([1]), processes of the intermediate type in which the excape of the faster nucleons is favoured by a non-equilibrium system.

The way things stood, it seemed of interest to check whether in the reactions described in *A*), to which the evaporation model appeared to represent a good approximation, the behaviour of the cross-sections of the various reactions agreed with this model. It is pointed out that as far as the α particles are concerned, Lassen ([5]) and co-workers, have, in the few cases measured, shown how this agreement was well established.

The aim of the present work is to show how the ratio $\sigma(\text{np})/\sigma(\text{n}, \text{n}_\text{I})$, with the 14 MeV neutrons (*) is well represented by the evaporation model for nuclei whose Z is between 10 and 40.

([14]) P. C. Gugelot: *Phys. Rev.*, **93**, 425 (1954).

([15]) R. Britten: *Phys. Rev.*, **88**, 283 (1952).

([16]) G. Igo: *Phys. Rev.*, **106**, 256 (1957).

([17]) B. L. Cohen: *Phys. Rev.*, **116**, 426 (1959).

([18]) L. Colli, I. Iori, M. G. Marcazzan, F. Merzari, A. M. Sona and P. G. Sona: *Nuovo Cimento*, to be published.

(*) By $\sigma(\text{n}, \text{n}_\text{I})$ we intend the total cross-section for the emission of the first neutron, that is $\sigma(\text{n}, \text{n}_\text{I}) = \sigma(\text{n}, \text{n}') + \sigma(\text{n}, 2\text{n}) + \sigma(\text{n}, \text{np})$.

In particular, a choice was made of the parameters to introduce in the calculation of the model, and the value of the nuclear radius was conveniently adjusted.

With this radius established, the θ values from the experimental spectra themselves were afterwards obtained.

This model can also explain the great changes of σ(n, np) and σ(n, 2n) between one nucleus and another, observed by various authors and discussed in [1]. An appreciable agreement was also obtained for the cross-sections of (p, n) reactions at 12 MeV in those cases in which they are known.

2. – Spectra of protons from n, p and p, p′ reactions.

We shall discuss with greater detail the spectra of protons emitted in reactions n, p and p, p′ obtained by various authors, and refer to backward angles.

It is pointed out that only backward angles are considered, to avoid the presence of direct effects and of deutons that have often been counted as protons.

The first problem of interest is the choice of the values of the inverse cross-section σ^*. We did not take into account the fact of the emitting nucleus being excited, nor that in the calculation of the σ^* the fraction of inverse reactions that produces direct effects should be treated separately, as we presumed that these effects do not give considerable deviations.

As is well known, the most recent views on the nuclear potential are derived from the optical model giving to the real part of the potential the shape of a well with rounded edges described by the formula [19] as a function of the r co-ordinate

$$V(r) = \frac{V_0}{1 + \exp[(r - R)/a]},$$

where: $V_0 \simeq 50$ MeV,
$R \simeq 1.3\,A^{\frac{1}{3}}\cdot 10^{-13}$ cm,
$a \simeq 0.5 \quad 10^{-13}$ cm.

The term of Coulomb barrier is associated with this nuclear potential.

Hitherto, the calculations of the Coulomb barrier penetrability available refer to a rectangular sharp edged well [11]. EVANS [20] has therefore shown how in the case of Sn the penetrability corresponding to the correct shape of the potential is equivalent to that given by a rectangular well with radius.

$$R = 1.62\,A^{\frac{1}{3}}\cdot 10^{-13}\ \text{cm}.$$

[19] H. FESHBACH: *Ann. Rev. Nucl. Sic.*, **8**, 49 (1958).
[20] J. A. EVANS: *Proc. Phys. Soc.*, **73**, 33 (1958).

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This radius, which we may call the radius of the Coulomb barrier represents a fictitious rectangular well which intersects the Coulomb potential approximately at the distance corresponding to the correct barrier height.

In this way, the height of the Coulomb barrier, as well as its penetrability, can be determined nearly correctly.

To obtain a better analysis of the experimental data we took into consideration the Coulomb barrier transparencies relative to three values of the r_0: 1.5; 1.6; $1.7 \cdot 10^{-13}$ cm.

The various spectra of n, p reactions therefore considered were divided by the corresponding $\varepsilon\sigma^*$.

The various curves $n(\varepsilon)/\varepsilon\sigma^*$ have a shape that by a rough approximation can be governed by an exponential law $\exp[-\varepsilon/\theta]$ at least up to a certain length (Fig. 1); accordingly, the curves proportional to the density of the nuclear levels can be obtained, as they are given by $\exp[E/\theta]$.

The values of θ corresponding to the various measurements and to the three values of r_0 are listed in Table I, where the value intervals of ε and E, to which the θ values refer, are also indicated.

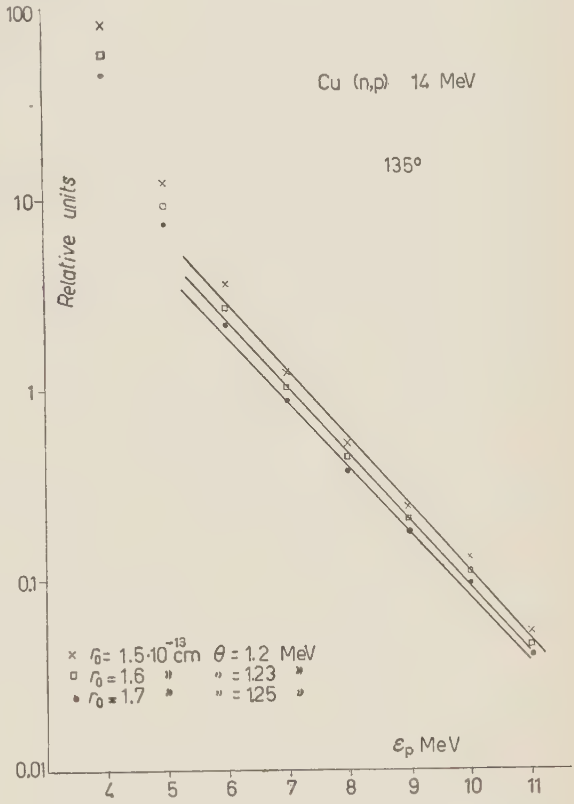

Fig. 1. – Values of θ calculated from the backward spectra of the (n, p) reaction on Cu at 14 MeV [1]. σ^* is calculated from the tables of Weisskopf, Feshbach and Shapiro [11] with values of $r_0 = 1.5$, 1.6 and 1.7 times 10^{-13} cm. The three curves are roughly represented with $\exp[-\varepsilon/\theta]$ with the indicated θ values.

θ is not found very sensitive to the value of the radius r_0: it varies from 1 to 1.2 MeV in the various cases for medium nuclei, and is of the order of 1.45 MeV for ^{32}S. The values of E for which these considerations are valid, range from $(2 \div 3)$ MeV to $(8 \div 10)$ MeV.

Apparently from the examination of the curves $n/\varepsilon\sigma^*$ it is not established whether the law of the dependence of $\omega(E)$ on E is better represented by the

TABLE I.

Nucleus	Reaction	Incident energy (MeV)	Range of emission energy (MeV)	Range of excitation energy (MeV)	Values of θ (MeV) $r_0=1.5$	$r_0=1.6$	$r_0=1.7$	Reference
^{32}S	np	14	$4\div 8$	$5\div 9$	1.4	1.43	1.46	(*a*)
^{40}Ca	np	14	$4\div 10$	$3.5\div 9.5$	1.02	1.05	1.07	(*b*)
Fe	pp′	11.3	$4\div 10$	$1.5\div 7.5$	1.30	1.35	1.42	(*c*)
^{54}Fe	np	14	$5\div 10$	$4\div 9$	1.15	1.18	1.20	(*d*)
^{56}Fe	np	14	$3\div 9$	$2\div 8$	1.22	1.25	1.28	(*e*)
Ni	np	14	$5\div 11$	$2\div 8$	1.20	1.22	1.23	(*b*)
^{58}Ni	np	14	$5\div 10$	$4\div 9$	1.0	1.02	1.05	(*f*)
^{60}Ni	np	14	$4\div 9$	$3\div 8$	1.1	1.15	1.17	(*e*)
Cu	pp′	11.3	$4\div 9$	$2.5\div 7.5$	1.2	1.25	1.30	(*c*)
Cu	np	14	$5\div 11$	$4\div 10$	1.2	1.23	1.26	(*g*)

(*a*) L. COLLI, I. IORI, M. G. MARCAZZAN, F. MERZARI, A. M. SONA and P. G. SONA: to be published in *Nuovo Cimento*.
(*b*) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN, M. PIGNANELLI and A. M. SONA: *Nuovo Cimento*, **7**, 400 (1958).
(*c*) B. L. COHEN and A. G. RUBIN: *Phys. Rev.*, **113**, 579 (1958).
(*d*) D. L. ALLAN: *Nucl. Phys.*, **10**, 348 (1959).
(*e*) P. V. MARCH and W. T. MORTON: *Phil. Mag.*, **3**, 143 (1959).
(*f*) I. KUMABE and R. W. FINK: *Nucl. Phys.*, **15**, 316 (1960).
(*g*) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN and A. M. SONA: *Nuovo Cimento*, **13**, 730 (1959).

exponential $\exp[E/\theta]$ or by a law of the type $\exp[\sqrt{aE}]$ in agreement with the model which requires the nucleus to be represented by a Fermi gas [12] (see [7] and [8]).

In fact, the interval of values for which $\omega(E)$ is obtained, is too small to enable us to choose between the two forms. It would be necessary to know $\omega(E)$ on a larger interval of E values, and with greater care. Often, in fact, the shape of the curve $\omega(E)$ is changed by contributive causes due to direct effects which can even be found at backward angles and particularly at high emission energy.

Favouring a simple exponential law is the analysis made by ERICSON [13] for values of E between 2 and $(5 \div 6)$ MeV on a great number of levels of some medium Z nuclei. Table I [13] is in fact obtained for θ:

^{28}Al: 1.6 MeV; ^{33}S: 1.6 MeV; ^{56}Mn: 1.15 MeV; ^{55}Fe, ^{57}Fe, ^{58}Fe: 1.2 MeV.

These values of θ as has already been pointed out correspond to those of Table I.

Furthermore, the spectra of neutrons obtained from nn′ reactions at 14 MeV [9,21] have more or less a Maxwellian shape with $\theta \simeq 1$ MeV, and seem to point to the validity of the exponential law up to values between 10 and 12 MeV for E.

More recent studies on nn′ reactions at 4 and 7 MeV [22] seem to point to slightly lower values of θ for energies E of a few MeV.

At low emission energies ($(3 \div 4)$ MeV) the proton spectra cannot generally be used for the calculation of $\omega(E)$. At these energies, in fact, secondary protons emitted in nnp reactions are often present, as has been demonstrated by ALLAN [23] and other authors [1].

At these energies, the curves $n(\varepsilon)/\sigma^*\varepsilon$ show a sharp rise versus E, which can more or less be represented by an exponential $\exp[\alpha E]$ with $\alpha \sim 2$ MeV^{-1}. On account of this behaviour, distinctly different, the secondary protons can easily be subtracted from the main spectrum.

At these low energies, moreover, the values of σ^*, are critically dependent on the Coulomb barrier used, and distortions can therefore occur due to the approximation applied [*].

In conclusion, we may affirm that in calculating the ratio $\sigma(\mathrm{n, p})/\sigma(\mathrm{n, n_I})$ it can be assumed that in the range of interest, the law $\omega(E)$, is represented, at least in a rough approximation, by the exponential law $\exp[E/\theta]$, the values

[21] E. R. GRAVES and L. ROSEN: *Phys. Rev.*, **89**, 343 (1953).

[22] D. B. THOMPSON and L. CRANBERG: *Bull. Am. Phys. Soc.* (April 30, 1959), p. 258.

[23] D. L. ALLAN: *Proc. Phys. Soc.*, A **70**, 195 (1957).

[*] In studying the n, p spectrum on Zn with neutrons of 7 MeV obtained by ROSEN [6], as also the pp′ spectra of COHEN and RUBIN [7] obtained at 11.3 and 13.5 MeV for values of ε corresponding to lower emission energies ($(2 \div 3)$ MeV), it should be noted that the curves undergo an appreciable displacement from the usual exponential of pendency: $\theta \approx (1 \div 1.2)$ MeV. This displacement results in greater steepness of the curve which reminds us of the effect of secondary protons observed earlier above.

Inversely, in these reactions the energy is too low to produce protons due to nnp or pnp reactions. The hypothesis that this displacement is due to insufficient knowledge of σ^* is supported by the fact that in the three examined cases the displacement occurs not by reason of a E determined value but rather by reason of different values of E in the three cases and a determined value of ε around 3 MeV.

of θ being as indicated in Table I. In fact, the parts of greater energy of the spectral areas where direct effects could alter the shapes of $\omega(E)$, and those of less energy where the nnp reactions and other effects make the interpretation difficult, are not important for the purpose of calculating the ratio $\sigma(\text{n, p})/\sigma(\text{n, n}_\text{I})$.

3. – Cross section of n, p reactions.

Cross-sections of n, p reactions have been measured by various authors in the case of twenty six nuclei in the Z area in which we are interested.

Some results were obtained by measuring directly the emitted protons, others from activation measurements of the residual nucleus.

We shall now sum up the conclusions of various authors with respect to the measurement of emitted protons.

MARCH and MORTON (4), COLLI and co-workers (1,24), and ALLAN (3) point to the presence of more components in the spectra. With respect to angular distribution, some of the emitted particles are almost isotropic, while others show a forward asimmetry.

The authors attribute these to direct emission, while the isothropic particles to statistic emission. Recently, COLLI and co-workers (25) were able to demonstrate how often in previous measurements deutons due to n, d reactions and practically present in all the studied nuclei were counted as protons. These deutons, actually, are mostly emitted forward and do not, therefore, contribute to the isotropic part of the n, p reaction (26). In studying the spectra of n, p reactions in which deutons were excluded, COLLI and co-workers (27) have, in a recent work, brought into evidence the existence, for light nuclei with Z between 12 and 40, of well-defined groups of protons whose angular distribution proved to be strongly forward (18). These groups could be clearly attributed to processes of collision of the instantaneous type.

As we are interested in the isotropic part of the n, p spectra, we can point out the following:

1) The isotropic part is dominant in all the cases studied; in fact the asymmetric part, whether due to n, p reactions of the direct type or to n, d reactions, do not exceed 10 to 20% of the total.

(24) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN, A. PIGNANELLI and A. M. SONA: *Nuovo Cimento*, **7**, 400 (1958).

(25) L. COLLI, F. CVELBAR, S. MICHELETTI and M. PIGNANELLI: *Nuovo Cimento*, **13**, 868 (1959); **14**, 1220 (1959).

(26) L. COLLI, M. G. MARCAZZAN, F. MERZARI, P. G. SONA and F. TONOLINI: to be published in *Nuovo Cimento*.

(27) L. COLLI, F. CVELBAR, S. MICHELETTI and M. PIGNANELLI: *Nuovo Cimento*, **14**, 81 (1959).

2) The values given by the various authors for the isotropic part can be taken as $\sigma(n, p)$ related to the evaporative process.

It should be observed again that in these measurements the protons due to n, np reactions [23] are subtracted.

In this type of measurement, the $\sigma(n, p)$ includes eventual $\sigma(np, n)$ in which a proton comes out as a first particle.

The activation cross-sections have been deduced from Table III by RIBE [28] and from a recent work by L. I. PREISS and R. W. FINK [29], while the older measurements with large errors were excluded.

In this case, the $\sigma(np)$ includes both direct and statistic effects, which cannot be separated.

The following procedure was adopted:

1) For light nuclei, the direct effect, as has been said, is only a small fraction of the total [18-27].

2) For medium nuclei Z between 20 and 40, recent results obtained on the n, p spectrum at forward angles without the presence of deutons, show spectra of uniform behaviour, in which the direct effect seems [30] not important.

3) Eventual $\sigma(n, pn)$ is excluded from this emission, but it may be presumed not considerable.

It can therefore be assumed with good approximation that the $\sigma(np)$ obtained from activation represent also the $\sigma(n, p)$ of the statistic processes.

In the case of medium Z nuclei ($20 \div 40$) the $\sigma(n, \alpha)$ and $\sigma(n, d)$ are very small; we can consider the exit $\sigma(nn_I)$ of a neutron, given in a rough approximation, from

$$\sigma(n, n_I) = \sigma_{anel} - \sigma(n, p) .$$

For light nuclei the (n, α) cross-section have been taken into account appromately.

By utilizing the σ_{anel} obtained by McGREGOR and co-workers [31] for interpolation, the values of $\sigma(n, n_I)$ of Table II and the corresponding ratio $\sigma(n, p)/\sigma(n, n_I)$ are obtained.

([28]) F. L. RIBE: *Fast Neutron Physics*: to be published.
([29]) I. L. PREISS and R. W. FINK: *Nucl. Phys.*, **15**, 326 (1960).
([30]) L. COLLI, S. MICHELETTI and M. PIGNANELLI: to be published.
([31]) M. H. MACGREGOR, W. P. BALL and R. BOOTH: *Phys. Rev.*, **108**, 726 (1957).

TABLE II.

Nucleus	σ(n, p) (mb)	Reference	$\sigma(\text{n}, \text{n}_l)$ (mb)	$\frac{\sigma(\text{n},\text{p})}{\sigma(\text{n},\text{n}_l)}$	θ (MeV)	$\frac{\sigma(\text{n},\text{p})}{\sigma(\text{n},\text{n}_l)} \cdot \frac{1}{F_0(Z)}$	Q (MeV)	Δ_c (MeV)	Q'	Δ
^{24}Mg	109 (*)	(*a*)	627	0.17	1.45	0.69	− 4.73	+ 4.58	− 0.15	4.2
^{27}Al	66	(*b*, *c*, *d*, *e*, *f*)	824	0.08	1.45	0.34	− 1.81	+ 0.07	− 1.74	+ 0.22
^{28}Si	292	(*g*)	618	0.47	1.45	2.24	− 3.87	+ 4.30	+ 0.43	+ 5
^{31}P	88	(*c*, *h*)	882	0.1	1.45	0.53	− 0.70	− 0.39	− 1.09	− 0.2
^{32}S	285 (*)	(*a*)	662	0.43	1.45	2.52	− 0.93	+ 3.28	+ 2.35	+ 2.25
^{40}Ca	518 (*)	(*m*)	690	0.75	1.2	9.00	− 0.5	+ 3.23	+ 2.73	+ 3.12
^{46}Ti	240	(*n*)	1060	0.225	1.2	3.30	− 1.6	+ 3.14	+ 1.54	+ 3
^{47}Ti	170	(*n*)	1140	0.15	1.2	2.20	+ 0.1	+ 0.44	+ 0.54	+ 0.8
^{48}Ti	61	(*n*)	1234	0.05	1.2	0.70	− 3.2	+ 3.02	− 0.18	+ 2.76
^{49}Ti	33	(*o*)	1 307	0.025	1.2	0.37	− 1.2	+ 0.26	− 0.94	0
^{52}Cr	87	(*g*)	1290	0.067	1.2	1.20	− 3.1	+ 2.91	− 0.19	+ 3.3
^{54}Fe	350 (*)	(*p*, *q*)	1000	0.35	1.2	7.56	+ 0.16	+ 2.92	+ 3.08	+ 2.14
^{56}Fe	95 (*)	(*p*, *c*, *e*)	1305	0.073	1.2	1.56	− 2.9	+ 2.77	− 0.13	+ 3.4
^{59}Co	82	(*r*)	1380	0.06	1.2	1.44	− 0.78	+ 0.01	− 0.77	+ 1.2
^{58}Ni	265 (*)	(*r*, *s*)	1170	0.23	1.2	6.13	+ 0.60	+ 2.69	+ 3.29	+ 1.6
^{60}Ni	87 (*)	(*t*)	1317	[illegible]	1.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

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^{63}Cu	94 (*)	(*q*, *u*)	1390	0.067	1.2	1.97	+ 0.70	+ 0.07	+ 0.77	+ 0.06
^{65}Cu	19	(*e*)	1510	0.012	1.2	0.35	— 1.20	+ 0.09	— 1.21	+ 0.06
^{64}Zn	233	(*r*, *v*, *w*)	1290	0.181	1.2	5.84	+ 0.20	+ 2.53	+ 2.73	+ 1.88
^{66}Zn	75	(*o*, *e*, *w*, *r*)	1445	0.051	1.2	1.04	1.80	+ 2.55	+ 0.75	+ 2.4
^{67}Zn	53	(*o*)	1500	0.035	1.2	1.13	+ 0.20	— 0.43	— 0.23	— 0.06
^{70}Ge	93	(*w*)	1490	0.062	1.2	2.48	— 0.86	+ 2.88	+ 2.02	+ 1.96
^{72}Ge	34	(*w*, *o*)	1565	0.022	1.2	0.88	3.20	+ 2.87	0.33	+ 3.01
^{73}Ge	21	(*w*, *o*)	1595	0.013	1.2	0.52	— 0.60	— 0.11	— 0.71	— 0.16
^{76}As	19	(*n*)	1610	0.012	1.2	0.52	— 0.35	+ 0.11	— 0.24	— 0.45

(*) The direct effect has been subtracted.

Δ_c: values obtained from CAMERON ([32]). Δ: values obtained from Figg. 3 and 4. Q' is calculated with the Δ_c values.

(*a*) L. COLLI, I. IORI, M. G. MARCAZZAN, F. MERZARI, A. M. SONA and P. G. SONA: to be published in *Nuovo Cimento*.
(*b*) E. B. PAUL and R. L. CLARKE: *Can. Journ. Phys.*, **31**, 267 (1953).
(*c*) S. G. FORBES: *Phys. Rev.*, **88**, 1309 (1952).
(*d*) P. V. MARCH and W. T. MORTON: *Phil. Mag.*, **3**, 1256 (1958).
(*e*) S. YASUMI: *Journ. Phys. Soc. Japan*, **12**, 443 (1957).
(*f*) J. DEPROZ, G. LEGROS and R. SOLIN: *Colloque de Phys. Nucl.* (Grenoble, 1960).
(*g*) W. E. THOMPSON, J. M. FERGUSSON and B. D. KERN: *Bull. Am. Phys. Soc.*, **3**, 210 (1958).
(*h*) J. A. GRUNDL, R. L. HENKEL and B. L. PERKINS: *Phys. Rev.*, **109**, 425 (1958).
(*m*) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN, A. M. PIGNANELLI and A. M. SONA: *Nuovo Cimento*, **7**, 400 (1958).
(*n*) Quoted in RIBE: *Fast Neutron Physics*, to be published.
(*o*) V. N. LEVKOVSKI: *Sov. Phys. Journ. Exp. Theor. Phys.*, **6**, 1174 (1958).
(*p*) P. V. MARCH and W. T. MORTON: *Phil. Mag.*, **3**, 143 (1959).
(*q*) D. L. ALLAN: *Nucl. Phys.*, **10**, 348 (1959).
(*r*) I. L. PREISS and R. W. FINK: *Nucl. Phys.*, **15**, 326 (1960).
(*s*) I. KUMABE and R. W. FINK: *Nucl. Phys.*, **15**, 316 (1960).
(*t*) P. V. MARCH and W. T. MORTON: *Phil. Mag.*, **3**, 577 (1958).
(*u*) L. COLLI, U. FACCHINI, I. IORI, M. G. MARCAZZAN and A. M. SONA: *Nuovo Cimento*, **13**, 730 (1959).
(*v*) A. H. ARMSTRONG and L. ROSEN: to be published.
(*w*) H. G. BLOSSER: cited in RIBE ([28]).

The evaporative model allows the calculation of this relation according to the formula

$$(1)\qquad \frac{\sigma(\mathrm{n},\,\mathrm{p})}{\sigma(\mathrm{n},\,\mathrm{n_I})}=\frac{\displaystyle\int_0^{\varepsilon_{\mathrm{p\,max}}}\varepsilon_{\mathrm{p}}\sigma_{\mathrm{p}}^{*}\omega(E_{\mathrm{p}})\,\mathrm{d}\varepsilon_{\mathrm{p}}}{\displaystyle\int_0^{\varepsilon_{\mathrm{n\,max}}}\varepsilon_{\mathrm{n}}\sigma_{\mathrm{n}}^{*}\omega(E_{\mathrm{n}})\,\mathrm{d}\varepsilon_{\mathrm{n}}}\,.$$

with obvious sense of symbols.

In this formula $\varepsilon_{\max}=\varepsilon_0+Q$, ε_0 being the incident energy and Q the energetic tonality of the np reaction.

We can then assume

$$(2)\qquad \omega(E)=\exp\left[E/\theta\right]\qquad\qquad E \text{ being} = \varepsilon_0+Q'$$

and Q' being given by $Q+\Delta$; Δ is the difference between the pairing energy of the two residual nuclei due to the np and nn′ reactions.

It is to remember that the pairing energy is due to the coupling of nucleons and appears when Z or N are even.

If δ_{n} and δ_{p} are the values of pairing energy relative to even N or Z, we can easily obtain:

1) for nuclei with A even

$$\Delta=-\,\delta_{\mathrm{p}}-\delta_{\mathrm{n}}\;;$$

2) for nuclei with Z even and N odd

$$\Delta=-\,\delta_{\mathrm{p}}+\delta_{\mathrm{n}}\;;$$

3) for nuclei with Z odd an N even:

$$\Delta=+\,\delta_{\mathrm{p}}-\delta_{\mathrm{n}}\;.$$

The various δ refer to the residual nuclei.

It should be remembered that the values of δ have been calculated by CAMERON [32] to be of the order of $(1\div2)$ MeV for the various nuclei under consideration.

With these positions the formula (1) becomes:

$$(3)\qquad \frac{\sigma(\mathrm{n},\,\mathrm{p})}{\sigma(\mathrm{n},\,\mathrm{n_I})}=\exp\left[Q'/\theta\right]\cdot F_0(Z)\,,$$

[32] J. CAMERON: *Can. Journ. Phys.*, **36**, 1040 (1958).

where

$$F_0(Z) = \frac{\int\limits_0^{\varepsilon_0} \varepsilon_p \sigma_p^* \exp\left[-\varepsilon_p/\theta\right] d\varepsilon_p}{\int\limits_0^{\varepsilon_0} \varepsilon_n \sigma_n^* \exp\left[-\varepsilon_n/\theta\right] d\varepsilon_n}$$

With a rough approximation, it was assumed in the case of odd A, for the sake of simplicity:

$$\delta_n = \delta_p$$

that is,

$$\frac{\sigma(n, p)}{\sigma(n, n_I)} = \exp\left[Q/\theta\right]\cdot F_0(Z) \,. \tag{4}$$

In the expression $F_0(Z)$, ε_{max} was simply replaced by ε_0, since for such reactions ε_{max} is approximate to ε_0, and the replacement does not modify the results.

In calculating σ_n^* we have utilized the data of BLATT and WEISSKOPF (2) according to which $r_0 = 1.4\cdot 10^{-13}$, and the Q values of the WAPSTRA tables (33).

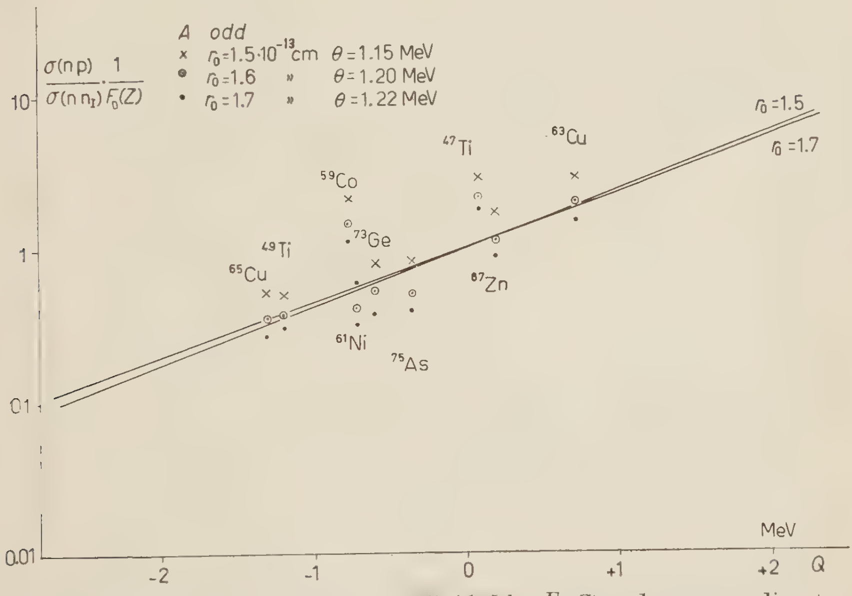

Fig. 2. – Odd A; values of $\sigma(n, p)/\sigma(n, n_I)$ divided by $F_0(Z)$ and corresponding to values of $r_0 = 1.5$; 1.6; 1.7 (10^{-13} cm). The lines give the expected values following eq. (4): $r_0 = 1.6$ seems to be the best value.

(33) A. H. WAPSTRA: *Physica*, **21**, 378 (1955).

For θ and for nuclei with Z between 20 and 30 were assumed the average values of Table I relative to the three values of r_0. These values differ very little from one another. Values of $F_0(Z)$ corresponding to these three values of r_0 were calculated, and Fig. 2 reports the values of $\sigma(\mathrm{np})/\sigma(\mathrm{nn_I})F_0$, as function of Q for the various odd nuclei.

The straight lines drawn are given by $\exp[Q/\theta]$, and it is noted that most of the points are distributed around such lines.

The value $r_0 = 1.6$ is the one that reconcile best the results, ^{57}Co and ^{47}Ti disagreeing by a factor of about 2.

A first study of the data relative to nuclei with A even, was made without taking into account the term Δ.

The radius $r_0 = 1.6$ having been used, Fig. 3 reports the values of $(\sigma(\mathrm{np})/\sigma(\mathrm{nn_I})\cdot(1/F_0(Z))$ for medium nuclei both even and odd and Fig. 4, the same values for lighter nuclei, for which θ is assumed to be 1.45 MeV.

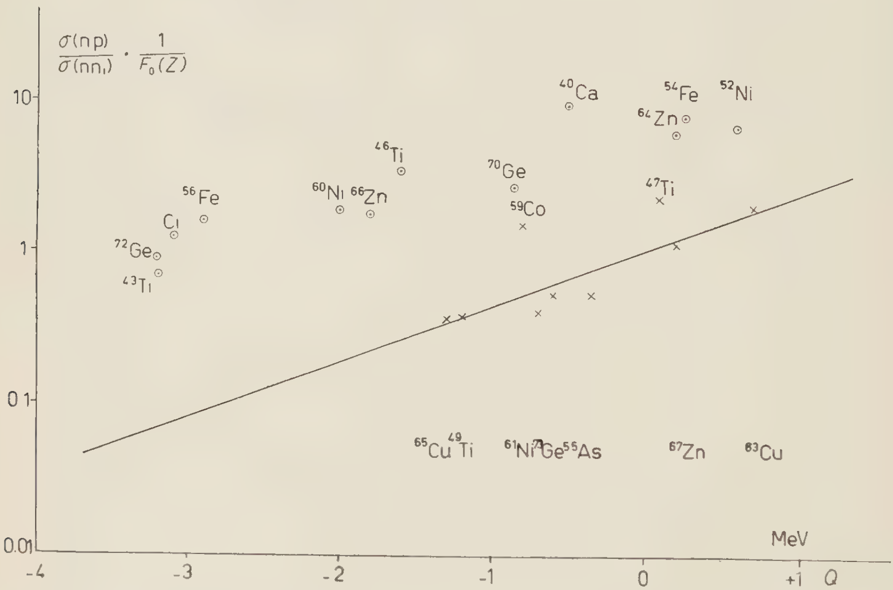

Fig. 3. – Even and odd A with $20 < Z < 40$, $\theta = 1.2$ MeV; $r_0 = 1.6\cdot 10^{-13}$ cm. Even nuclei are about $(2.5 \div 3)$ MeV left of the expected line given by $\exp[Q/\theta]$. x: odd A; ⊙: even A.

A horizontal displacement of the points corresponding to even nuclei results. This displacement marks the necessity of taking into account the value $\Delta = |\delta_p + \delta_n|$, according to formula (3). In fact, the displacement should be exactly equal to Δ itself.

For even nuclei, Fig. 3 and 4 give therefore the values of $|\delta_p + \delta_n|$ relative to the bombarded nucleus, and for odd nuclei the values of $|\delta_p - \delta_n|$.

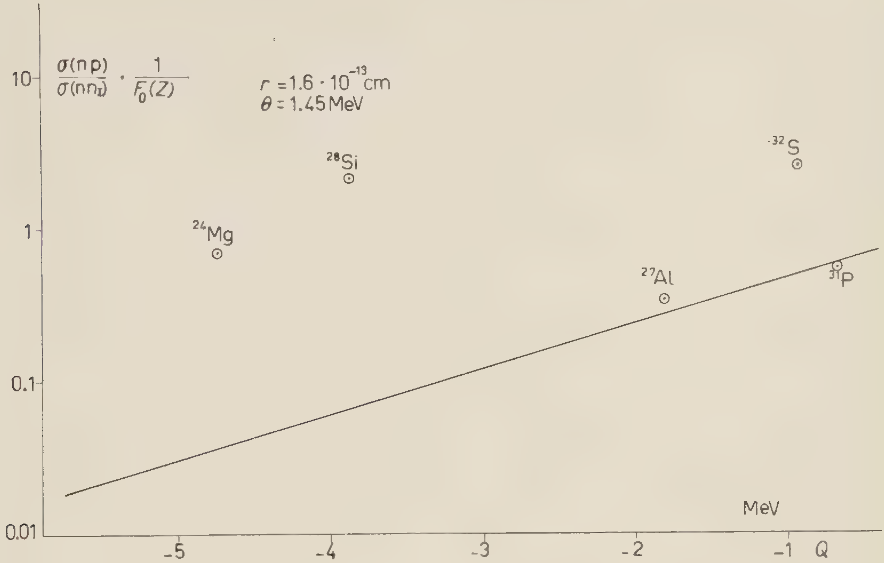

Fig. 4. – As in Fig. 3 but $12 < Z < 18$. θ has been chosen $= 1.45$ MeV.

It is possible to compare these values of Δ drawn up in column 10 of Table II and those obtained from the values δ_p and δ_n given by CAMERON ([32]) also shown in Table II.

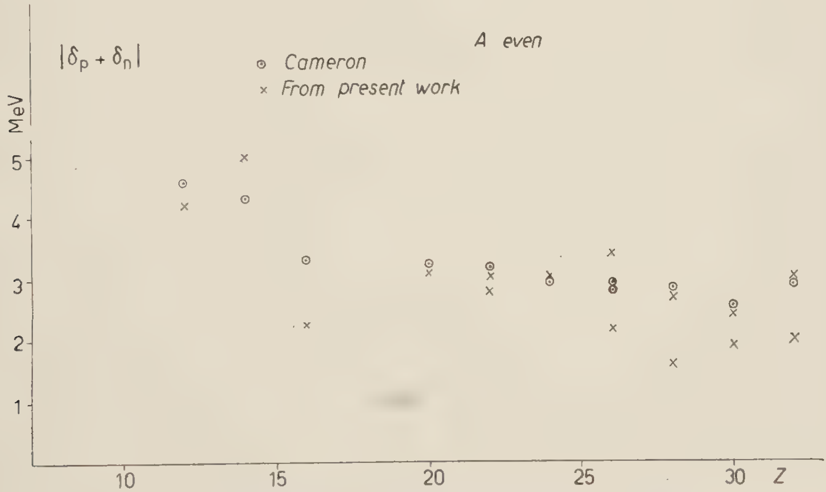

Fig. 5. – A even; the values of $\Delta = |\delta_p + \delta_n|$ obtained from Fig. 3 are compared with Δ obtained from Cameron values.

In spite of the various simplifications introduced, the comparison results in an excellent agreement.

In Fig. 5 are plotted the values of $|\delta_p+\delta_n|$ obtained by CAMERON for even nuclei and those obtained from Figg. 3 and 4.

To conclude this discussion, we think it was useful to give again the $\sigma(n, p)/\sigma(n, n_I)$ by introducing the values obtained according to CAMERON. Fig. 6 expresses the values of $(\sigma(n, p)/\sigma(n, n_I)\cdot(1/F_0(Z))$ as function of Q'.

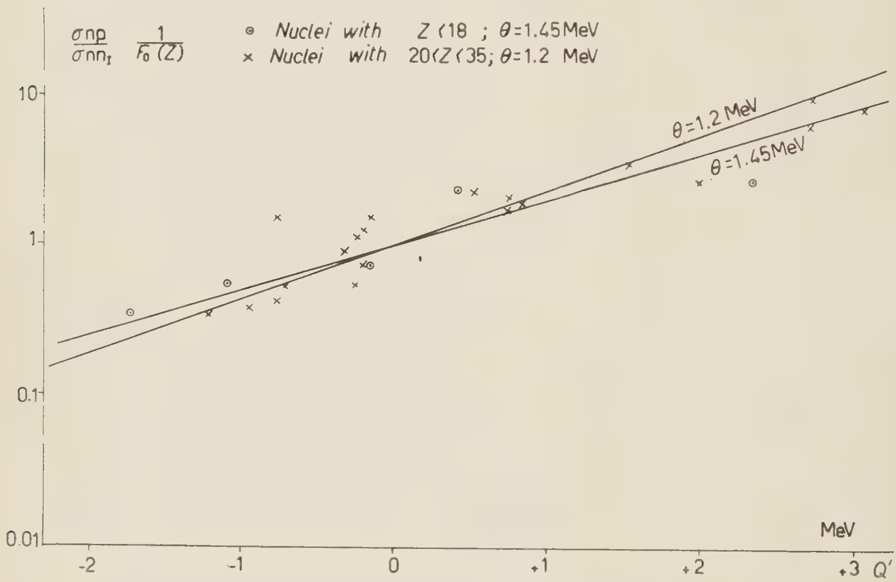

Fig. 6. – Values of $([\sigma(n, p)/\sigma(n, n_I)](1/F_0(Z)))$ for the nuclei of Fig. 3 ($\times$) and of Fig. 4 ($\odot$) plotted as functions of Q'. Light nuclei ($\odot$) are required to be on the line given by $\exp[Q'/\theta]$ with $\theta=1.45$ MeV, other nuclei correspond to $\theta=1.2$ MeV.

On the whole, we can say that with respect to these nuclei there is good evidence that the np and nn′ reactions occur according to the principle of equiprobability of the final states.

4. – n, 2n and n, np reactions at 14 MeV.

It would be interesting to examine the application of the evaporative model to the n, 2n and nnp reactions on medium nuclei.

Let A represent a medium nucleus. It has been shown [1] how in some medium nuclei the $\sigma(n, 2n)$ is smaller than it would be if the emission of the second neutron were the dominant process, where such emission is energetically possible, that is, where the energy of the first neutron is less

than $14+L_n$, L_n being the binding energy of the last neutron in the nucleus A. In this case, the excitation energy of A is $> |L_n|$, and A can emit the second neutron.

It has been shown that by indicating the average width of the total excitation of the so-excited nucleus A with Γ, and the emission width of the neutron with Γ_n, Γ_n/Γ is often less than 1.

It was also concluded in (1) that often, in strong competiton with the second neutron, a proton is emitted according to the n, np reaction in spite of the effect of the Coulomb barrier that reduces the emission of protons, which have energy smaller than the height of the barrier itself.

TABLE III.

Nucleus	$-L_n+L_p$ (MeV)	Δ (MeV)	$\exp\left[-\dfrac{L_n+L_p+\Delta}{\theta}\right]$	$F(Z)$	$\left[\dfrac{\Gamma_p}{\Gamma_n}\right]_{calc.}$	$\left[\dfrac{\Gamma_p}{\Gamma_n}\right]_{exp.}$
^{56}Fe	1.02	0.13	2.61	0.009	0.022	< 0.04
^{58}Ni	4	0.05	29	0.05	1.45	3.8
^{60}Ni	1.9	− 0.09	4.5	0.014	0.061	0.1
^{63}Cu	4.72	− 2.81	4.9	0.04	0.2	0.31
^{65}Cu	1.47	− 2.83	0.32	0.026	0.008	< 0.04
^{64}Zn	4.1	− 0.35	22.7	0.043	0.98	1.13
^{90}Zr	3.66	− 0.27	16.8	0.0075	0.126	0?
^{92}Mo	4.35	− 0.11	34	0.011	0.37	1.3

All the values $(\Gamma_p/\Gamma_n)_{exp}$ are taken from (1), but the ^{64}Zn is taken from (6).

The ratio σ(n, np)/σ(n, 2n) cannot be taken as an indication of the emission ratio between protons and neutrons by nucleus A, because the secondary protons can also be emitted by less-excited nuclei A, for which the emission of the neutron is not energetically possible. In fact the σ(n, np) is due to all these nuclei from which the first neutron has been emitted with energy $< 14+L_p$, where L_p is the binding energy of the proton. Since $|L_p|$ is smaller than $|L_n|$, a part of the protons are emitted by excited nuclei at energies E, comprised between $|L_p| < E < |L_n|$, and only a fraction of the σ(n, np) is directly in competition with the σ(n, 2n).

We have estimated with an approximate reasoning (more than anything else, we are actually interested in the order of magnitude) the fraction f of the σ(n, np) than can be assumed to be in competition with the σ(n, 2n), and in Table VII of [1] we gave the values of Γ_p/Γ_n, which for convenience are transcribed in Table IV. Though roughly estimated, these values show great variation between one nucleus and another.

TABLE IV.

Nucleus	$\sigma(p, n)_{(exp)}$ (mb)	Reference	σ_p (*) (mb)	$Q'(pp') - Q'(pn)$ (MeV)	$Q'(p\alpha) - Q'(pn)$ (MeV)	$\dfrac{\Gamma_p}{\Gamma_n}$	$\dfrac{\Gamma_\alpha}{\Gamma_n}$	$\sigma(p, n)_{(calc)}$ (mb)	$\dfrac{\sigma(p, n)_{(exp)}}{\sigma(p, n)_{(calc)}}$
^{45}Sc	350 ± 130	(*a*)	+760	+3,05	+3.62	0.96	0.19	353	1
^{52}Cr	285 ± 55	(*a*)	+790	+2.69	+1.81	0.55	0.03	500	0.57
^{60}Ni	370 ± 65	(*a*)	+820	+3.97	+5.58	1.03	0.36	343	1.08
^{61}Ni	590 ± 160	(*a*)	+820	+2.89	+4.28	0.41	0.12	530	1.1
^{63}Cu	525	(*b*)	+825	+3.75	+5.96	0.77	0.43	375	1.4
^{65}Cu	830	(*b*)	+825	+1.73	+5.08	0.14	0.21	611	1.35
^{64}Zn	390	(*b*)	+830	+5.27	+7.74	2.74	1.7	152	2.56
^{66}Zn	612	(*a*, *b*)	+830	+3.35	+6.09	0.52	0.39	434	1.41
^{68}Zn	950	(*a*, *b*)	+830	+0.99	+4.34	0.07	0.09	715	1.32
^{89}Y	750 ± 100	(*a*)	+840	+3.33	+3.77	0.21	0.007	688	1.09

(*a*) H. G. BLOSSER and T. H. HANDLEY: *Phys. Rev.*, **100**, 1340 (1955).
(*b*) H. A. HOWE: *Phys. Rev.*, **109**, 2083 (1958).

As we wish to apply the evaporative model to these emissions, we should point out that:

1) When the final levels are few, as often happens in these cases since E is only a little greater than L_n, the statistic model may not represent a sufficient approximation.

2) The use of the function $\omega(E)$ is in such a case a great simplification.

3) The low energy of the emitted protons shows that the values of σ_p^* are more critically dependent on the chosen value of r_0. Therefore, now to the

purpose of showing that the great variations observed in the ratio Γ_p/Γ_n can conform to the evaporation model, rough calculations were made, and we assumed that the excitation energy of nucleus A was fixed at 13 MeV.

In this case, which does not change the substance of the calculation, the exit of the first neutron corresponds to a fixed energy of 1 MeV.

We have assumed that the law $\exp[E/\theta]$ was valid for the first few MeV. Thus

$$\frac{\Gamma_p}{\Gamma_n} = \frac{\int_0^{\varepsilon_{p\,max}} \varepsilon_p \sigma_p^* \exp[E_p/\theta]\,d\varepsilon_p}{\int_0^{\varepsilon_{n\,max}} \varepsilon_n \sigma_n^* \exp[E_n/\theta]\,d\varepsilon_n},$$

where:

$$\begin{aligned} \varepsilon_{p\,max} &= 13 + L_p\,,\\ \varepsilon_{n\,max} &= 13 + L_n\,,\\ E_p &= 13 + L'_p - \varepsilon_p\,,\\ E_n &= 13 + L'_n - \varepsilon_n\,, \end{aligned}$$

where L'_p and L'_n correspond to L_n and L_p plus the pairing energy of the correspondent residual nucleus. We have assumed here the average value of $\theta = 1.2$ MeV, the Q values are obtained from Wapstra tables [33] and the pairing energies from CAMERON [32]; r_0 is $1.6 \cdot 10^{-13}$ cm for protons and $1.4 \cdot 10^{-13}$ cm for neutrons.

Thus we have

$$\frac{\Gamma_p}{\Gamma_n} = \exp[(L'_p - L'_n)/\theta] \cdot F(Z)\,,$$

where

$$F(Z) = \frac{\int_0^{\varepsilon_{p\,max}} \varepsilon_p \sigma_p^* \exp[-\varepsilon_p/\theta]\,d\varepsilon_p}{\int_0^{\varepsilon_{n\,max}} \varepsilon_n \sigma_n^* \exp[-\varepsilon_n/\theta]\,d\varepsilon_n}.$$

Evidently, from this calculation too, we can obtain only an estimate of the ratio Γ_p/Γ_n, which can, however, be compared with the values obtained from experiment (Table IV).

Apart from the case of ^{90}Zr, which is not clear, the parallel behaviour of experimental and calculated values is strongly indicative of the validity of the model which, though simplified, takes into account the actions of factors $30 \div 50$ from one nucleus to another.

It is interesting to note how the high value of the expression $\dfrac{\exp[L'_p/\theta]}{\exp[L'_n/\theta]}$ in some nuclei compensates the reduction in proton emission produced by the barrier, thus bringing the value of Γ_p/Γ_n to the order of unity. Fig. 7 plots the values of $(\Gamma_p/\Gamma_n)_{\exp}(1/F(Z))$ as a function of $L'_p - L'_n$.

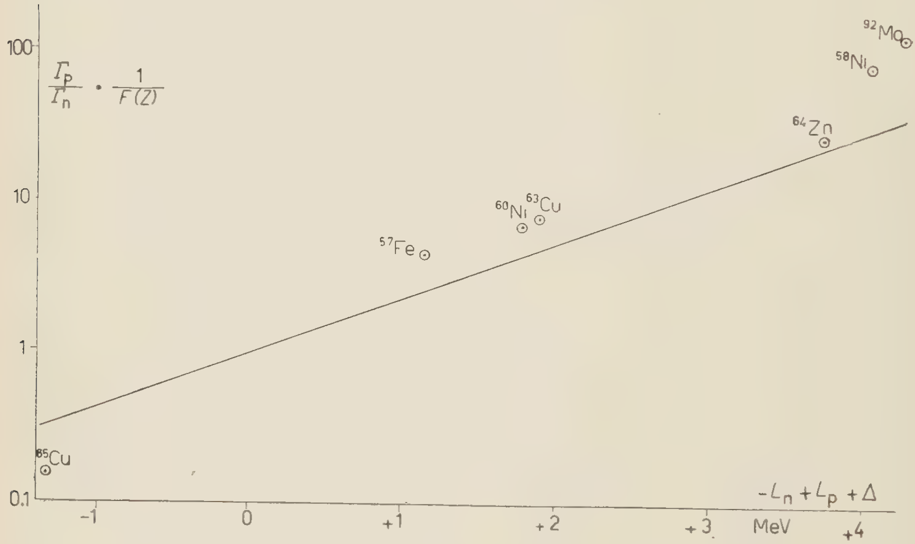

Fig. 7. – Values of Γ_p/Γ_n deduced from n, np and n, 2n reaction as function of $-L_n + L_p$. The line gives the expected behaviour.

5. – p, n reactions at 12 MeV.

Few values of σ(p, n) are known, and we have chosen those at 12 MeV; various complications interfere at greater energies (direct effects, multiple reactions, etc.).

The values used are those of BLOSSER ([34]), excluding those with great errors, and those of HOVE ([35]), as indicated in Table V.

([34]) H. G. BLOSSER and T. H. HANDLEY: *Phys. Rev.*, **100**, 1340 (1955).
([35]) H. A. HOWE: *Phys. Rev.*, **109**, 2083 (1958).

The evaporative calculation is made by using the formula

$$\sigma(\mathrm{p},\mathrm{n}) = \sigma_\mathrm{p}^* \frac{\Gamma_\mathrm{n}}{\Gamma_\mathrm{n} + \Gamma_\mathrm{p} + \Gamma_\alpha}\,. \tag{6}$$

where:

$$\Gamma_\mathrm{n} = \exp\,[Q'_\mathrm{n}/\theta]\int_0^{\varepsilon_{\max \mathrm{n}}} \sigma_\mathrm{n}^*\varepsilon_\mathrm{n} \exp\,[-\,\varepsilon_\mathrm{n}/\theta]\,\mathrm{d}\varepsilon_\mathrm{n}\,,$$

$$\Gamma_\mathrm{p} = \exp\,[Q'_\mathrm{p}/\theta]\int_0^{\varepsilon_{\max \mathrm{p}}} \sigma_\mathrm{p}^*\varepsilon_\mathrm{p} \exp\,[-\,\varepsilon_\mathrm{p}/\theta]\,\mathrm{d}\varepsilon_\mathrm{p}\,,$$

$$\Gamma_\alpha = 2\exp\,[Q'_\alpha/\theta]\int_0^{\varepsilon_{\max \alpha}} \sigma_\alpha^*\varepsilon_\alpha \exp\,[-\,\varepsilon_\alpha/\theta]\,\mathrm{d}\varepsilon_\alpha\,,$$

with evident meaning of the symbols.

The radii parameters used are for protons $r_0 = 1.6\cdot 10^{-13}$ cm, for neutrons $r_0 = 1.4\cdot 10^{-13}$ cm and for α particles, the radii used by LASSEN [8]. θ values are 1.2 MeV.

The table contains the values of σ_p^*, $\Gamma_\mathrm{p}/\Gamma_\mathrm{n}$ and $\Gamma_\alpha/\Gamma_\mathrm{n}$ besides those of $\sigma(\mathrm{p},\mathrm{n})$ calculated according to (6). The agreement is good.

It is interesting to note how often the p, n reaction is not the dominant reaction, in view of Q being favourable to p, p′ and p, α reactions which compete strongly with it.

6. – Conclusion.

Along with the direct effects of various nature observed in recent years, the present work shows the presence of some groups of reactions, in which the emission of protons and neutrons of medium and light nuclei is regulated by the law of equiprobability which forms the basis of Weisskopf's evaporative model. These conclusions are supported by the agreement between some thirty calculated and experimental cross-sections regarding the emission of protons and neutrons in np, nn′, n, np, n 2n and pn reactions in medium nuclei. The figures used for σ^* and for $\omega(E)$ correspond to the more recent views on the Coulomb barrier shape and on the level densities in medium nuclei.

The evaporative process is important for not-too-high incidental energies: $(13 \div 18)$ MeV according to the various reactions.

For higher Z nuclei, it is well known that the emission of neutrons is the dominant one, and it can be safely said that this emission is mostly evaporative.

The direct effects seem, on the other hand, responsible for the small emission of protons in heavy nuclei ($Z > 50$) (*). Direct effects become dominant, also, at greater energies than those indicated above for all the types of reactions and nuclei.

Returning to medium Z nuclei we recall:

1) Some of the hypotheses used in the past concerning the evaporative model, and particularly the one according to which the emission of neutrons is the dominant process in the p, n or α, n reactions, or for n, 2n reactions concerning the second neutron, should be discarded. In fact, the drop in the emission of protons and particles due to Coulomb barrier is often compensated by favourable Q' values.

2) A particularly good agreement was observed between the calculated and measured $\sigma(\mathrm{n, p})/\sigma(\mathrm{n, n_I})$. The trend ratio brings well into evidence the dependence of the reaction on the properties of the residual nuclei: dependence on Q and the fact that the residual nuclei are even even or odd odd.

From the study of the even nuclei it is possible to obtain the values $\delta_{\mathrm{p}} + \delta_{\mathrm{n}}$, but, the measurements are not sufficiently exact or extensive to give a complete table of the various figures. It is also to be expected that a small difference between the behaviour of one nucleus and that of another interfere in the reaction.

3) The spectra of the evaporative reactions make it possible to check the density curves of the levels $\omega(E)$. The presence of non-evaporative emissions, the simplifications used in calculating σ^*, the presence of multiple reactions etc., limit the accuracy of the information obtained. The law according to which it depends on E is not clear, therefore, and a valid rough approximation in a limited interval of E values indicates that it is well represented by an exponential $\exp[E/\theta]$ with θ values of the order of 1.45 MeV for nuclei with Z between 10 and 20, and 1.2 MeV for nuclei with Z between 20 and 40.

Even in such a case it is not possible to know the changes in θ from one nucleus to another because of the scarce material available.

4) We may conclude by observing that the great variations in the cross-sections n, 2n and nnp observed for some medium nuclei can be adequately combined under an evaporative model. Variations in $\Gamma_{\mathrm{p}}/\Gamma_{\mathrm{n}}$ between 0.01 and some units come particularly under this scheme.

(*) A calculation with the evaporative model of $\sigma(\mathrm{n, p})/\sigma(\mathrm{n, n_I})$ for these nuclei shows how the experimental value is much greater than that of the calculation.

RIASSUNTO

È stato analizzato un certo numero di reazioni nucleari per nuclei medi e leggeri ($10 < Z < 40$) ed in particolare reazioni (n, p) prodotte da neutroni di 14 MeV. La espressione $n(\varepsilon)/\varepsilon\sigma^*$ che è proporzionale alla densità dei livelli nucleari è abbastanza bene rappresentata per valori di ε compresi tra 4 e 10 MeV, da una legge esponenziale del tipo $e^{-\varepsilon/\theta}$ con $\theta=(1\div1.2)$ MeV per nuclei medi con Z fra 20 e 40, e $\theta=1.45$ MeV per nuclei più leggeri. La σ^* è stata calcolata assumendo per i neutroni $r_0=1.4\cdot10^{-13}$ cm e per i protoni $r_0=1.6\cdot10^{-13}$ cm. In questa ipotesi si ha che il rapporto $\sigma(\mathrm{n,p})/\sigma(\mathrm{n,n_I})$ è in buon accordo col modello evaporativo, quando si tenga conto del valore di Q della reazione e del fatto che il nucleo residuo sia pari-pari o dispari-dispari. Da tale rapporto, calcolato per i nuclei pari, si ottengono i valori della «pairing energy» Δ in buon accordo con quelli calcolati da Cameron. Anche nel caso della reazione (n, 2n), (n, np) e (p, n) per nuclei medi, il modello evaporativo spiega in modo soddisfacente le variazioni dei valori delle sezioni d'urto.

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Angular Dependence of Polarization in p-p Scattering at 970 MeV. Preliminary Results.

R. J. Homer, G. W. Hutchinson, W. K. McFarlane, A. W. O'Dell, R. Rubinstein and E. J. Sacharidis

Department of Physics, University of Birmingham - Birmingham

(ricevuto il 3 Marzo 1960)

We are investigating the angular dependence of the asymmetry in proton-proton elastic scattering using a partially polarised beam of 970 MeV protons. At this energy the F wave contribution to elastic p-p scattering is expected to be considerable. It is therefore of interest to see whether there is interference between P-wave and components of higher angular momentum in the spin-orbit interaction. If the spin-orbit splitting of the phase shifts are of the same sign, one would expect a reduction, or change of sign, of the asymmetry at large angles, close to 90° in the centre of mass system.

A proton beam is produced by scattering to the right at 4°÷5° from a carbon target inside the synchrotron. It is therefore polarised but the degree of polarisation is not well known. The most probable value is about $\frac{1}{3}$ (1). It is focussed and collimated and passes into a liquid hydrogen target. Elastic events are recognised in the second scattering by detecting both the scattered and recoil particles in coincidence.

The results so far obtained are shown in Fig. 1. The errors shown are from counting statistics only and represent one standard deviation. It appears that the sign of the asymmetry changes, not at larger angles, but at an angle less than 45° in the centre of mass system.

We are not able, at present, to investigate the behaviour at smaller angles because the efficiency for the detection of the recoil protons is already falling off seriously at the smallest angles we have used. This is illustrated in Fig. 2 in which the sum of the counting rates for right and left scatters is shown as a function of laboratory scattering angle. The change of slope of this curve could give rise to serious distortion of the asymmetry curve if the zero angle had been wrongly estimated. Fig. 3 shows a visual best fit to the experimental points and the way in which the results

(1) N. E. Booth, F. L. Hereford, M. Huq, G. W. Hutchinson, M. E. Law, A. M. Segar and D. H. White: *Proc. of the International Conf. on High Energy Physics* (Geneva, 1958).

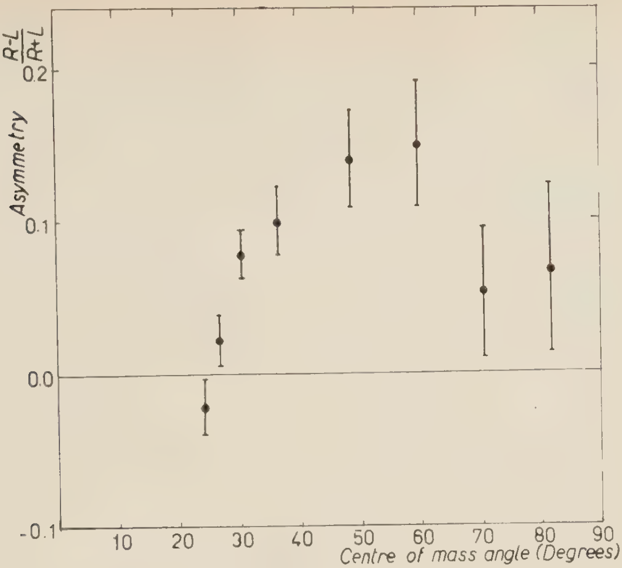

Fig. 1.

would be distorted by errors of $\frac{1}{10}°$ and $\frac{1}{2}°$ in estimating the central direction.

The zero angles was found by swinging each counting arm in turn through the unscattered proton beam, of greatly reduced intensity, and plotting the vari-

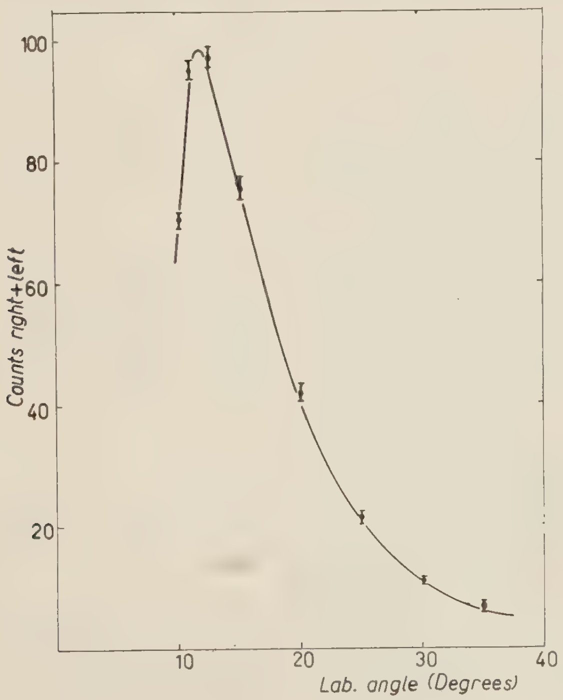

Fig. 2.

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ation of counting rate with angle. It is estimated that the error of the setting may be about $\pm\frac{1}{10}°$. This is confirmed by the requirement that the asymmetry should be zero at 90° in the centre of mass. We thus have considerable confidence in the conclusion that the asymmetry becomes small at a centre of mass angle close to 25°.

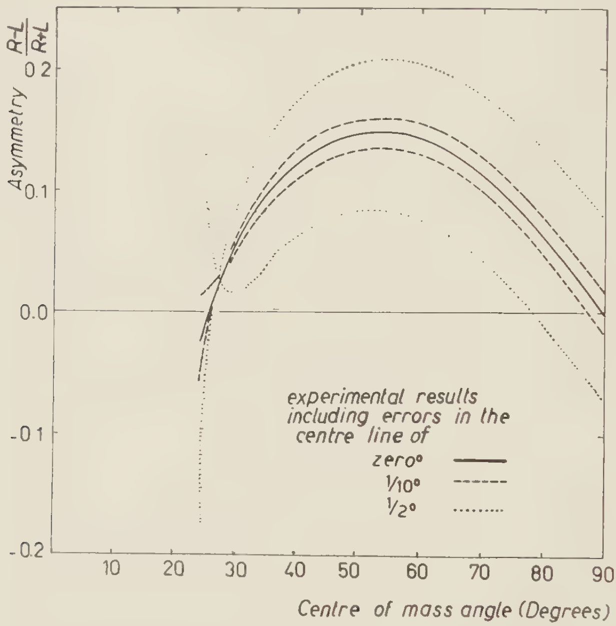

Fig. 3.

The most obvious interpretation of this result, in a non-relativistic approximation in which the p-p interaction is represented by a potential, is that the spin-orbit splitting term of the interaction changes sign at a distance a little less than that corresponding to the classical impact parameter for F waves at 970 MeV, *i.e.* at about one Fermi.

It may be noted that experiments on p-p polarisation at 635 MeV [2] showed angular variations which would be compatible with a reduction of the asymmetry at small angles by F wave interference. The F wave contribution at this energy would be expected to be smaller, and confined to smaller angles, than at 970 MeV.

We are at present modifying our apparatus to make it possible for us to investigate the asymmetry at smaller angles. We hope also to reduce our statistical errors at larger angles and to publish a fuller account of the work later.

* * *

We are greatly indebted to Professor R. E. PEIERLS, Professor G. E. BROWN and Mr. D. SPRUNG for discussions and advice on the interpretation of our results; to the Synchrotron team for the operation of the machine and to Professor P. B. MOON for support and encouragement.

(2) M. G. MESHCHERIAKOV, S. B. NURUSHEV and O. G. STOLETOV: *Žurn. Ėksp. Teor. Fiz.*, 33, 37 (1957). Translated in *Sov. Phys.* (*Journ. Exp. Theor. Phys.*), 6, 28 (1958).

IL NUOVO CIMENTO Vol. XVI, N. 6 16 Giugno 1960

Preliminary Results on the Interactions in Photographic Emulsion of K⁻-Mesons at 1.15 GeV/c.

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University College - Dublin

(ricevuto il 16 Marzo 1960)

A stack of 100 stripped emulsions, each $17.5 \times 12.5 \times 0.06$ cm³, was exposed to a separated K⁻-beam at $(1.15 \pm \pm 0.02)$ GeV/c momentum at the Bevatron. During one third of the exposure time the stack was placed in the direct beam, and for the remainder it was placed behind the 15 in. hydrogen bubble chamber. The beam composition, determined by scanning the bubble chamber pictures, has been published as [1]

$$4.5\ \mu^- : 1.5\ K^- : 0.2\ \pi^- .$$

Preliminary results on two other stacks, exposed under somewhat similar conditions as the present one, have been given by Barkas *et al.* [2] and Garelli *et al.* [3].

(*) Present address: Lawrence Radiation Laboratory, Berkeley.

(1) L. W. Alvarez P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho and S. G. Wojcicki: *Phys. Rev. Lett.*, **2**, 215 (1959).

(2) W. H. Barkas, N. N. Biswas, D. A. de Lise, J. N. Dyer, H. H. Heckman and F. M. Smith: *Phys. Rev. Lett.*, **2**, 466 (1959).

(3) C. M. Garelli, B. Quassiati, L. Tallone and M. Vigone: *Nuovo Cimento*, **13**, 1294 (1959).

1. – Results.

Two methods of detecting K⁻-interactions were used—track scanning and area scanning. In the track scan 84.66 m of minimum track in the beam direction were followed, and 144 stars observed. The mean free path for the particle mixture of the beam is thus (59 ± 5) cm, and if the beam composition of [1] is used this implies a mean free path for K⁻-mesons in emulsion of (17 ± 2) cm (geometric m.f.p. ≈ 25 cm). There is no bias against very small stars in track scanning so that the prong distribution $(N_h + n_s)$ and the shower track $(g \leqslant 1.5 g_{\min})$ distribution (n_s) in Fig. 1 are believed to represent the true characteristics of the stars produced by the interacting component of the beam.

In the area scanning the areas chosen consisted of two strips transverse to the beam 1 cm wide 10.5 cm long, spaced 13.5 cm apart measured in the beam direction. In an area of 209 cm² 138 stars with primaries within 30° of the beam direction were recorded. The prong distributions of these, given in Fig. 2, are markedly different from Fig. 1, showing a very small number of 0-, 1- and 2-prong

stars, which are presumably biassed against by this method of scanning. Fig. 1 indicates that such small stars

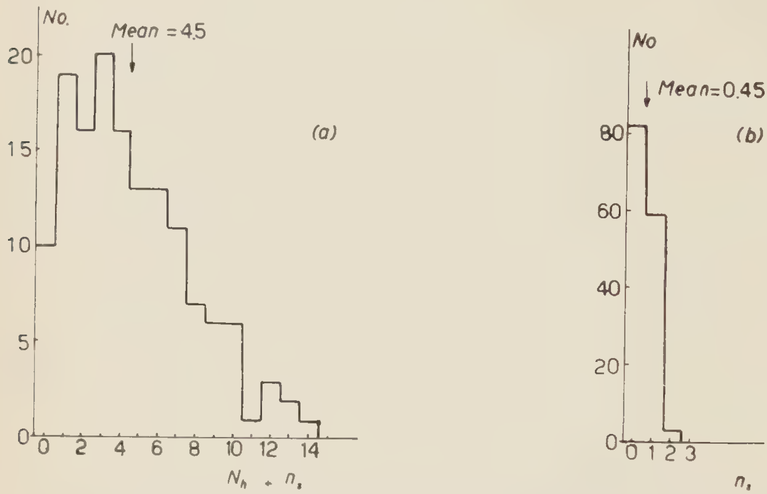

Fig. 1. – Prong number distribution for stars found by track scanning. *a*) Total prong number distribution. *b*) Shower prong number distribution.

constitute 31% of all stars. On results obtained by area scanning only BARKAS *et al.* [2] claim that only a small fraction of stars have $\leqslant 2$ prongs (about 3%).

At the present stage of the analysis unbiassed results are available from a sample of 107 track scanned stars. Assuming the beam composition quoted, 8 of these should have been caused by π^--mesons. All tracks with $g/g_{\min} \geqslant 1.2$ have been followed until they interacted, decayed, ended or left the Dub. portion of the stack. Sixteen emergent K-mesons

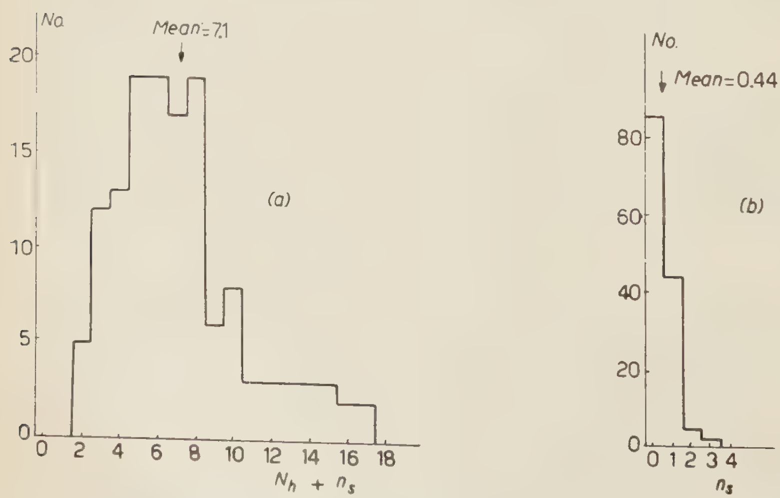

Fig. 2. – Prong number distribution for stars found by area scanning. *a*) Total prong number distribution. *b*) Shower prongs number ditribution.

have been observed, 8 of which ended and are negatively charged; their energy distribution is shown in Fig. 3. This result is in contrast to the behavior at low energies (($\sim 100 \div 200$) MeV) where

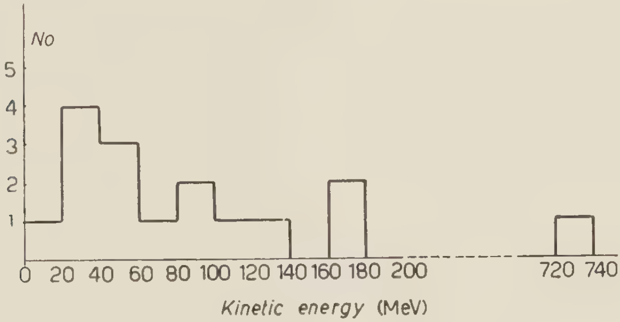

Fig. 3. – Energy distribution of K-mesons emitted from 107 analysed stars.

an interacting K^--meson is very rarely seen to emerge. In four cases the inelastically scattered K^--meson created a π-meson. Eight $\Sigma^{\pm}$-hyperons were observed, none is consistent with having been a Ξ^--particle. Thus a lower limit for strange particle emission is 22%. It is estimated that single π-meson production occurs in 22% of cases and double π-meson production in 1%. No case of double hyperon production has been observed.

Eleven 2-prong events, which show no sign of a recoil nucleus or β-emission, have been analysed, and K-mesons emerge from 4 of these. From kinematical arguments three are examples of the processes

$$\begin{aligned} K^- + H &\rightarrow K^- + p \\ &\rightarrow \pi^- + \Sigma^+ \\ &\rightarrow K^- + \pi^+ + n \end{aligned}$$

respectively.

2. – Mean free path.

Measurements are in progress to determine independently the beam composition in the stack and see if the figures of (1) are indeed applicable to this exposure. If they are then the cross-section for K^--mesons, by track scanning, is anomalously high (~ 1.5 geometric). If the μ-meson contamination is substantially less than is thought then the mean free path could be geometric.

In the area scanning the two strips were scanned alternately in sequence to avoid any slowly changing scanning bias. The number of stars with primaries within 30° of the beam direction was 95 in Strip 1 (2 cm from beam entry edge) and 43 in Strip 2 (2 cm from beam exit edge). The ratio of star densities in the two strips is a measure of the attenuation mean free path of the $(\pi+K)$-component of the beam, and unlike the track scanning figures is independent of the μ-contamination. In taking the star densities of the two strips it is necessary to decide on the angular cut-off of the primaries of stars to be accepted, as the angular distribution of interacting particles after passing through the stack is considerably broadened due to diffraction scattering.

Accepting all stars with primaries within 5° of the beam direction in Strip 1 and within 20° in Strip 2 the mean free path for $(\pi+K)$ for star production turns out to be $\leqslant 23$ cm. This value is regarded as an upper limit, since the cut-off angle of 20° should allow the inclusion of all diffraction scattered beam particles.

* * *

We are grateful to Dr. E. J. LOFGREN and the Bevatron crew for permission and the facilities to expose at the Bevatron, and to Dr. D. J. PROWSE for his care and attention during the exposure of the stack.

IL NUOVO CIMENTO Vol. XVI, N. 6 16 Giugno 1960

On the Substitution Law in Quantum Field Theory.

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(ricevuto il 19 Aprile 1960)

The validity of the substitution law, proposed by Jauch and Rohrlich ([1,2]) for quantum electrodynamics, is examined for quantum field theory in general, without recourse to perturbation expansion. A general condition on the interaction Hamiltonian is obtained which is sufficient to ensure that the S-matrix satisfies this law.

In this note we shall only summarize our discussion.

In considering elementary particle processes, it is usually assumed that if the process

$$\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots\rightarrow\alpha+\beta+\gamma+\ldots \tag{1}$$

is possible, then another process, the step-conjugate process

$$\mathrm{b}+\mathrm{c}+\ldots\rightarrow\bar{\mathrm{a}}+\alpha+\beta+\ldots \tag{2}$$

is also possible, where $\bar{\mathrm{a}}$ is the anti-particle of a. The substitution law makes a quantitative statement about these related processes ([1]). Let the 4-momenta of the particles a, b, ..., α, β, ... participating in (1) be k_a, k_b, ..., k_α, k_β, ..., and let the S-matrix elements for the processes (1) and (2) be

$$M(k_\alpha, k_\beta, \ldots; k_\mathrm{a}, k_\mathrm{b}, \ldots), \qquad M(k_{\bar{\mathrm{a}}}, k_\mathrm{b}, \ldots; k_\mathrm{b}, k_\mathrm{c}, \ldots),$$

respectively. Then the substitution law states essentially that

$$M(k_{\bar{\mathrm{a}}}, k_\alpha, \ldots; k_\mathrm{b}, k_\mathrm{c}, \ldots) = M(k_\alpha, k_\beta, \ldots; -k_\mathrm{a}, k_\mathrm{b}, \ldots), \tag{3}$$

where $-k_\mathrm{a}$ means k_a with the sign of all four components reversed.

([1]) J. M. Jauch and F. Rohrlich: *Theory of Photons and Electrons* (Cambridge, 1955).
([2]) F. Chew: *Ann. Rev. Nucl. Sci.*, **9**, 29 (1959).

To investigate the validity of eq. (3), we consider first the case of bosons. We assume that the free fields obey the ordinary, relativistically invariant, equations of motion and commutation relations (assumption (A)), and we adopt the interaction picture. We assume nothing about the relativistic invariance of the interaction, but we do assume that the interaction Hamiltonian density, $\mathscr{H}(x)$, is a function of the positive and negative frequency parts of the field operators and of their derivatives *of finite order only* (assumption (B)). Although we shall work with the interaction picture, the perturbation expansion is not used throughout.

We write the positive and negative frequency parts of the field operator $\varphi(x)$ in terms of particle-annihilation and anti-particle-creation operators for definite momentum by

$$(4)\qquad \begin{cases} \varphi^{(+)}(x) = \dfrac{1}{\sqrt{2}}\dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int\dfrac{\mathrm{d}^3\boldsymbol{k}}{\omega_{\boldsymbol{k}}}\,a(k)\exp[ikx]\,, \\ \varphi^{(-)}(x) = \dfrac{1}{\sqrt{2}}\dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int\dfrac{\mathrm{d}^3\boldsymbol{k}}{\omega_{\boldsymbol{k}}}\,b^{\dagger}(k)\exp[-ikx], \end{cases}$$

where $k=(\boldsymbol{k}, i\omega_{\boldsymbol{k}})$, $\omega_{\boldsymbol{k}}=+\sqrt{(\boldsymbol{k}^2+\mu^2)}$, with similar equations for the Hermitian conjugate operators. The $a(k)$ and $b(k)$ satisfy

$$[a(k), a^{\dagger}(k')] = \omega_{\boldsymbol{k}}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,.$$

Denoting both $a(k)$ and $b(k)$ indifferently by $d(k)$, the matrix element of an arbitrary operator R between two states of definite momentum is, apart from possible normalization factors

$$(5)\qquad \langle d(k_1') \dots d(k_n') R\, d^{\dagger}(k_1) \dots d^{\dagger}(k_m)\rangle_0\,.$$

This can be written as

$$(6)\qquad (-1)^n\langle[[\dots[R, d^{\dagger}(k_1)]d^{\dagger}(k_2)]\dots d^{\dagger}(k_m)]d(k_1')]\dots d(k_n')]\rangle_0 + \text{other terms}\,.$$

The «other terms» correspond to processes represented by disconnected Feynman graphs, and we disregard them. The multiple commutator is symmetric under any interchange of the d's and $d^{\dagger}$'s: hence it is sufficient to establish the relation

$$(7)\qquad [U, a^{\dagger}(k)]_{k\to -k} = -[U, b(k)]\,,$$

in order to prove the substitution law for the matrix element of U.

We define the transition matrix $U(\tau)$ by

$$(8)\qquad i\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) = H(\tau)\,U(\tau)\,, \qquad\qquad U(-\infty) = 1\,.$$

We can show that (7) is equivalent to

$$(9)\qquad [H(\tau), a^{\dagger}(k)]_{k\to -k} = -[H(\tau), b(k)]\,.$$

A *sufficient* condition (I) that (9) should hold (and one that is also necessary if $\mathscr{H}(x)$ contains no derivatives of the field operators) is that

$$\frac{\partial\mathscr{H}(x)}{\partial\varphi^{(+)}(x)}=\frac{\partial\mathscr{H}(x)}{\partial\varphi^{(-)}(x)}, \tag{10}$$

i.e. $\varphi^{(+)}(x)$ and $\varphi^{(-)}(x)$ enter $\mathscr{H}(x)$ *only in the combination*

$$\varphi(x)\equiv\varphi^{(+)}(x)+\varphi^{(-)}(x)\,.$$

The procedure is similar for the case of fermions. We make the further assumption that the fermion field functions occur only in even combinations (assumption (C)). The decomposition (6) is replaced by a similar one, with alternate anti-commutators and commutators. A certain difficulty arises in this fermion case, for it is not clear where exactly to the left of the operator in (5) one should put the operator $b(k)$, and a change in position will give a change in sign. If we accept this change as being unimportant, since it does not affect transition probabilities in any way, then the condition (I) (with, of course, the spinor field function $\psi(x)$ instead of $\varphi(x)$) is again sufficient for the holding of the substitution law, which now reads

$$M(k_{\bar{a}}, k_\alpha, \dots;\ k_b, k_c, \dots)=\pm M(k_\alpha, k_\beta, \dots;\ -k_a, k_b, \dots)\,. \tag{11}$$

We may notice that the condition (II): $\varphi^{(+)}(x)$ and $\varphi^{(-)}(x)$ enter $\mathscr{H}(x)$ only in the combination

$$\varphi^{(1)}(x)=i[\varphi^{(+)}(x)-\varphi^{(-)}(x)]\,,$$

is also sufficient for the « relaxed » law (11) to hold. The « crossing theorem » can be derived from either of the relations (3) or (11), and hence either (I) or (II) is sufficient for this theorem.

It is obvious from the above argument that the substitution law does not hold for Lee's model in which $\varphi^{(+)}$ and $\varphi^{(-)}$ are interacting separately. It is interesting to notice in this connection, that Ter-Martirosjan points out that the analyticity and the unitarity of the S-matrix in Lee's model are not compatible [3]. We may notice also that an operator which obeys our conditions on the Hamiltonian density is a local operator in the sense of Bogoliubov and Shirkov, and is thus « locally causal », in that two such operators at space-like separated points commute [4].

A detailed account will appear in the Proceedings of the Royal Irish Academy, Series A.

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[3] K. A. Ter-Martirosjan: *Žurn. Ėksp. Teor. Fiz.*, 37, 1005 (1959).

[4] N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (New York, 1959).

A Note on the Relative Parity $p_{K\Sigma}$.

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(ricevuto il 4 Maggio 1960)

Recently Capps (1) has suggested some interesting possibilities to determine the parity of the K-hyperon pair *i.e.* the relative parity p_{KY}. In this note we adopt his results to point out two more methods to decide the sign of $p_{K\Sigma}$. In the first method, we study the decay of polarized Σ^0. Though this has been studied (2) our calculation does not depend upon whether or not the polarization of the γ-ray is observed. Our second suggestion makes use of the reaction $K^- + p \to \Sigma^0 + \gamma$.

It has been shown (1) that if in the reaction $\pi^- + p \to \Sigma^0 + K^0$ the target protons are polarized perpendicular to the incident pion beam, then the Σ^0's emitted in the directions $\theta = 0$ or π (θ being measured from the direction of the incident pion beam) are polarized transversely, the polarization being given by

$$p_\Sigma(\mp, \boldsymbol{m}) = \pm\, p_i(\boldsymbol{m})\,, \tag{1}$$

where p_i denotes the degree of polarization of the target in the direction $\boldsymbol{m}$. The minus or plus sign in $p_\Sigma(\mp, \boldsymbol{m})$ correspond to the two cases, namely when $p_{K\Sigma}$ is odd or even respectively. In the subsequent decay of these polarized Σ^0's, if we observe the Λ^0's emitted in the direction $\boldsymbol{m}$ (in the rest system of Σ^0) the polarization of these Λ^0's is given by

$$p_\Lambda(\boldsymbol{m}) = -\, p_\Sigma(\mp, \boldsymbol{m})\,. \tag{2}$$

This result is independent of the Σ^0-Λ^0 relative parity.

Thus we have

$$\begin{cases} p_\Lambda(\boldsymbol{m}) = -\, p_i(\boldsymbol{m}) & \text{for } p_{K\Sigma} = \text{odd}, \\ p_\Lambda(\boldsymbol{m}) = +\, p_i(\boldsymbol{m}) & \text{for } p_{K\Sigma} = \text{even}, \end{cases} \tag{3}$$

i.e. the sign of the polarization is the same as that of $p_{K\Sigma}$. As has been noted (1), the direction of the polarization of Λ^0 can be deduced from the asymmetry of its decay products. However if we do not select the Λ^0's in any specific direction, we have (2)

$$p_\Lambda = -\tfrac{1}{3} p_\Sigma\,. \tag{4}$$

Hence

$$\begin{cases} p_\Lambda(\boldsymbol{m}) = -\tfrac{1}{3} p_i(\boldsymbol{m}) & \text{for } p_{K\Sigma} = \text{odd}, \\ p_\Lambda(\boldsymbol{m}) = +\tfrac{1}{3} p_i(\boldsymbol{m}) & \text{for } p_{K\Sigma} = \text{even}. \end{cases} \tag{5}$$

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(1) R. H. Capps: *Phys. Rev.*, **115**, 736 (1959).

(2) G. Feldman and T. Fulton: *Nucl. Phys.*, **8**, 106 (1958).

Even in this case the sign of the polarization is unaltered, though its magnitude is reduced.

Consider the reaction

$$K^- + p \to \Sigma^0 + \gamma \; . \tag{6}$$

Following the method in reference (1), we can discuss the polarization p_Σ when the target protons are polarized perpendicular to the incident K^- beam. We now find that even if $p_i = 0$ (*i.e.* unpolarized target)

$$\begin{cases} p_\Sigma \neq 0 & \text{for } p_{K\Sigma} = \text{even}, \\ p_\Sigma = 0 & \text{for } p_{K\Sigma} = \text{odd}: \end{cases} \tag{7}$$

To detect p_Σ, we make use of the first method by observing the decay of these polarized Σ^0. Perhaps the main defect of this method is that it is rather difficult to identify the reaction (6). We are at present working out the polarization correlation between the produced and decay (*i.e.* $\Sigma^0 \to \Lambda^0 + \gamma$) γ-rays.

* * *

We thank Professor ALLADI RAMAKRISHNAN and Mr. K. VENKATESAN for the benefit of very interesting discussions.

IL NUOVO CIMENTO Vol. XVI, N. 6 16 Giugno 1960

Validity of Q.E.D. in μ Pair Production.

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(ricevuto il 14 Maggio 1960)

Recently it was noticed (1) that the actual technical improvements make possible, using colliding beams, the realization of high energy experiments. Particularly all these experiments concern the interaction of electrons with electrons and positrons. Moreover, while electron-electron scattering happens only with the exchange of a photon (to e^2 order), electron-positron interaction can take place also via the annihilation and subsequent recreation of the electron positron pair. If the acting energy is sufficiently high (higher than a certain threshold) near the primitive process, the production of different particles, in number and conditions fixed by the various selection rules will become possible. Beside, calling p_+ and p_- the momenta of the positron and electron, the momentum transfer in the annihilation of the pair is $q^2 = (p_+ + p_-)^2$ and in the C.M.S. $q^2 = -4\varepsilon^2$, independent of the angle and far away from the mass-shell. So, we understand that this form of the interaction e^+-e^- is very much favourable for an inquiry on the validity at short distances of the fundamental conceptions of the actual quantum field theory. In fact, as proposed by many authors, electron-positron annihilation can give particularized information on the limits of validity of the Q.E.D. (2,3) and on the structure of the elementary particles.

For example Cabibbo and Gatto (4) have shown the possibility of measuring directly the photon-pion vertex (and the correspondent form factors) with processes of the type:

$$e^+ + e^- \rightarrow n \text{ pions}.$$

The object of this short note is to study the production of $\mu^+\mu^-$ pairs in the e^+e^- annihilation.

---

(1) W. K. H. Panofsky: reported at the *Ninth Annual International Conference on High-Energy Physics* (Kiev, 1959), unpublished.

(2) G. Furlan and G. Peressutti: *Nuovo Cimento*, **14**, 758 (1959).

(3) G. Andreassi, P. Budini and I. Reina: *Nuovo Cimento*, **12**, 488 (1959).

(4) N. Cabibbo and G. Gatto: *Phys. Rev. Lett.*, **4**, 313 (1960).

The presence of the μ-mesons should reveal favourable for an inquiry at small distances, because their Compton wave-length, less than that of the electron, is just of the linear dimensions of the regions that we wish to explore.

We assume that the μ-meson is not punctiform and that its interaction with the electromagnetic field is given by:

$$\Gamma_\mu = \gamma_\mu F_1(q^2) + \sigma_{\mu\nu} q_\nu F_2(q^2) \qquad \sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{2i} \,. \tag{1}$$

If the μ-meson doesn't allow other interactions different from the electromagnetic one, such a form of its vertex takes into account of a violation the Q.E.D. In the limits of a non-relativistic treatment we can identify F_1 with Fourier's transform of the charge distribution of the μ-meson. The second term, containing $F_2(q^2)$ means a different possibility of describing an alteration of the local electrodynamics and with it we can refer to an eventual magnetic structure of the μ-meson.

For the conventional electrodynamics $\Gamma_\mu = \gamma_\mu$ that is $F_1 = 1$, $F_2 = 0$ (or more exactly the term containing F_2 is $\neq 0$ for quantities of the order $\alpha = 1/137$ and contributes only to the radiative corrections). Together with the condition to vanish for large q^2, this requires that the two form factors satisfy the boundary conditions:

$$F_1(0) = 1 \qquad F_1(\infty) = 0 \,,$$

$$F_2(0) = 0 \qquad F_2(\infty) = 0 \,.$$

In the approximation to order e^2, the process is described by a diagram only and the cross-section gives easily the expression valid in every reference frame.

$$\begin{aligned} d\sigma = {} & \frac{\alpha^2}{4m^4(1+\varkappa)^2\sqrt{\varkappa^2-1}} \, \frac{\beta'^2_-\varepsilon'_-\, d\Omega'_-}{\beta'_- E - |p|\cos\theta'_-} \cdot \\ & \cdot \{2F_1^2(q^2)[(1+\varkappa)(M^2+m^2) + M^2(\mu^2+\lambda^2)] - \\ & - 4m^2F_1(q^2)F_2(q^2)\,(1+\varkappa)(2+\varkappa) + m^2F_2^2(q^2)(1+\varkappa)\,(3+\varkappa+2\mu\lambda)\} \,, \end{aligned} \tag{2}$$

where

$$E = \varepsilon_- + \varepsilon_+ \,, \qquad \boldsymbol{p} = \boldsymbol{p}_- + \boldsymbol{p}_+ \,,$$

θ'_- is the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'_-$; q^2 is the transfer impulse; μ, λ, k are invariants defined by:

$$\begin{aligned} -m^2 k &= p_- p_+ \,, \\ -mM\lambda &= p_+ p'_+ = p_- p'_- \,, \\ -mM\mu &= p_+ p'_- = p_- p'_+ \,, \end{aligned}$$

(*) *Note:* A form of the electromagnetic vertex equal to (I) has been used by AVAKOV (5) for an analysis of the validity of the Q.E.D. in the scattering e^--^{4}He.

(5) G. V. AVAKOV and K. A. TER-MARTIROSYAN: *Nucl. Phys.*, **13**, 685 (1959).

and by the energy momentum conservation

$$m(1+k) = M(\mu+\lambda)\,,$$

M is the mass of μ-meson, m is the mass of the electron.

It is convenient to particularize the expression in C.M.S. where we obtain with $m \sim 0$

$$\text{(3)}\qquad \sigma(\theta) = \frac{\alpha^2}{16\varepsilon^5}\sqrt{\varepsilon^2 - M^2}\,F_1^2(+4\varepsilon^2)\cdot$$
$$\cdot\left\{(\varepsilon^2+M^2)\left(1+\frac{\varepsilon^2}{M^2}\Delta^2\right)+(\varepsilon^2-M^2)\cos^2\theta\left(1-\frac{\varepsilon^2}{M^2}\Delta^2\right)-4\varepsilon^2\Delta\right\},$$

with $\Delta(q^2)=F_2(q^2)/F_1(q^2)$ and $\Delta(0)=0$, $\Delta(\infty)=1$, if we admit an analogous behaviour of F_1 and F_2 for large q. (These conditions are satisfied *e.g.* by $\Delta(q^2) = |q^2|/(\Lambda^2+|q^2|)$.)

Calling σ_0 the local cross-section ($F_1=1$, $F_2=0$) it follows by eq. (3) that the ratio $R=\sigma/\sigma_0$ can depend:

1) only on the energy if in $\sigma(\theta)$ $\Delta=0$;

2) on the energy and the angle if in $\sigma(\theta)$ $\Delta\neq 0$.

If such a dependence on the angle would be revealed experimentally, for $q^2=\text{const}$, that is for $\varepsilon=\text{const}$, it would give evidence of a break-down of Q.E.D. following also from the second term of (1).

As an indication in Figg. 1, 2 we plot $R=\sigma/\sigma_0F_1^2$ as a function of the scattering angle for fixed energies ($q^2=\text{const}$) and for the values $\Delta^2=0.1$; 0.2; 0.4; 0.6.

From Fig. 1 we see that the dependence of the ratio R on the scattering angle θ is such that the presence of the Δ term gives an effect of 10% for $\Delta^2=0.1$, passing for example from 90° ($\cos\theta=0$) to 24° ($\cos\theta=0.9$). The effect decreases with the increase of Δ^2 reaching a minimum after which we have an inversion of the behaviour of the function. Such an effect results more evident by increasing the energy ε (see plot Fig. 2).

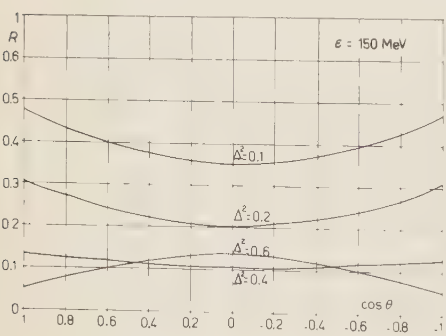

Fig. 1.

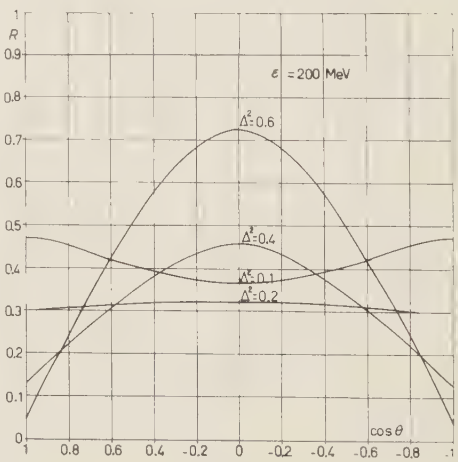

Fig. 2.

Lacking this behaviour the ratio σ/σ_0 measures directly the form factor $F_1(q^2)$, independent of the angle for q^2 fixed.

If, in the spirit of the idea of Drell [6], we assume for $F_1(q^2)$ the form $F_1(q^2)=$ $=\Lambda^2/(\Lambda^2+|q^2|)$, an experimental accuracy of 10% would make possible a lowering of the limits of validity of Q.E.D. till to $1/\Lambda \simeq 0.1F$ and $0.14F$, at 200 and 150 MeV respectively. On the other hand assuming, on the basis of the data of electron-proton scattering, as limit of validity of Q.E.D. $1/\Lambda=0.3$ fermi, we obtain at the same energies a non-local effect of 50% and 35%. (At such energies the total cross-section is of the order of $0.5\cdot10^{-30}$ cm^2).

So the studied process results to be very sensitive to a presence of a Drell's form factor or of a modification of the theory of the type proposed in (1) and would make it possible to test electrodynamics at distances of the order of 10^{-14} cm. The different behaviour as a function of the angle would give a way to discriminate between the two possibilities.

However all the results above proposed, have just an indicative value and we will be able to set up a definitive discussion only after the computation of the radiative corrections. This analysis will be discussed in a subsequent work.

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The authors wish to express their deep gratitude to Prof. P. Budini for his encouragement.

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(6) S. D. Drell: *Ann. Phys.*, **4**, 75 (1958).

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Remark on the τ^+ Meson Decay.

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(ricevuto il 31 Maggio 1960)

Several attempts have been carried out lately to explain the energy spectra and angular distributions of the decay products of the τ^+-meson,

$$\tau^+ \to \pi^- + \pi^+ + \pi^+ + 75 \text{ MeV} . \tag{1}$$

The experimental results of McKenna *et al.* (1) indicate a deviation of the energy spectra of positive and negative pions from the spectra predicted by DALITZ (2) for a spin zero K-meson. McKENNA *et al.* give:

$$\frac{\mathrm{d}w(\varepsilon_+)}{\mathrm{d}\varepsilon_+} = (1 + m_+\varepsilon_+)\sqrt{\varepsilon_+(1-\varepsilon_+)} , \qquad m_+ = -(0,214 \pm 0.026), \tag{2}$$

$$\frac{\mathrm{d}w(\varepsilon_-)}{\mathrm{d}\varepsilon_-} = (1 + m_-\varepsilon_-)\sqrt{\varepsilon_-(1-\varepsilon_-)} . \qquad m_- = (0.665 \pm 0.1): \tag{3}$$

ε_+ and ε_- being the kinetic energies of the positive and negative pions measured in units of their maximum kinetic energy $T_{\max} = \frac{2}{3}Q$, $\varepsilon_\pm = T_\pm / T_{\max}$, $T_\pm = p_\pm^2/2\mu$. The angular distribution on the other hand shows no significant deviation from isotropy. Its best fit being (3)

$$\frac{\mathrm{d}w(\theta)}{\mathrm{d}\cos\theta} = 1 - (0.037 \pm 0.072)\cos^2\theta , \tag{4}$$

θ being the angle between the directions of the relative motion of the two like pions $(\boldsymbol{p}_2 - \boldsymbol{p}_3)$ and the motion of the unlike pion $\boldsymbol{p}_1$.

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(1) S. McKENNA, S. NATALI, M. O'CONNELL, J. TIETGE and N. C. VARSHNEYA: *Nuovo Cimento*, 10, 763 (1958).

(2) R. H. DALITZ: *Phil. Mag.*, **44**, 1068 (1953); *Phys. Rev.*, **94**, 1046 (1954); E. FABRI: *Nuovo Cimento*, 11, 479 (1954).

(3) J. TIETGE: private ommunication.

This deviation has been explained by several authors as a consequence of $\pi\pi$ final state interaction, by HOLLADAY and THOMAS [4] using the Watson final state interaction technique and by KHURI and TREIMAN and SAWYER and WALI [5] by means of the Mandelstam representation technique. Both groups consider only S wave final states and fit the experimental results assuming a $\pi\pi$ scattering cross section of the order of $(100 \div 250)$ mb.

We want to expose here an alternative explanation which does not assume such a strong $\pi\pi$ interaction and which is essentially a variation of the Dalitz analysis [2]. We shall assume spin zero for the τ-meson and decompose its total angular momentum into the relative angular momenta of two unlike pions ($ij=13$ or 12) l_{ij} and of the third pion with respect to the center of mass of the first two l'_{ij}. The τ-meson spin being zero, $l_{ij}=l'_{ij}$. We write the partial wave development of the decay amplitude as:

$$(5)\qquad M=\sum_{l_{ij}=l'_{ij}}\ \sum_{\substack{(ij)=(12)\\(31)}} c^{(ij)}_{l_{ij}l'_{ij}}(p_{ij}, p'_{ij})\, C_{l_{ij}l'_{ij}}(S=0, S_z=0; 0, M=0)\sqrt{\frac{2l_{ij}+1}{4\pi}}\, Y^{M=0}_{l'_{ij}}(\theta_{ij}, \varphi_{ij})\,.$$

In formula (5) p_{ij} and p'_{ij} stand for the moduli of:

$$(6)\qquad \boldsymbol{p}_{ij}=\frac{1}{\sqrt{2}}(\boldsymbol{p}_i-\boldsymbol{p}_j)\,,\qquad \boldsymbol{p}'_{ij}=\sqrt{\frac{2}{3}}\left(\boldsymbol{p}_k-\frac{\boldsymbol{p}_i+\boldsymbol{p}_j}{2}\right),$$

θ_{ij} being the angle between $\boldsymbol{p}_j$ and $\boldsymbol{p}'_{ij}$.

The amplitude M must be symmetrical under the exchange of the two like mesons 2 and 3. That is:

$$(7)\qquad c^{(13)}_{ll'}(p, p') = c^{(12)}_{ll'}(p, p')\,.$$

Assuming the order of magnitude of coefficients $c^{(ij)}_{ll'}(p, p')$ to be determined by angular momenta barrier considerations, one expects them to be of the form:

$$(8)\qquad c^{(ij)}_{ll'}(p, p')\sim\frac{A^{(ij)}_{l+l'}}{(l+l'+4)!!}\cdot\frac{(l+1)!!}{(2l+1)!!}\cdot\frac{(l'+1)!!}{(2l'+1)!!}\left(\frac{p'}{Mc}\right)^{l}\left(\frac{p}{Mc}\right)^{l'}$$

where $A^{(ij)}_{l+l'}$ is a coefficient of slowly varying modulus and the radius of the K-meson is written: $R=\hbar/Mc$. For reasonable K-meson radii of the order of the π-meson Compton wave length, the ratio of D to S wave contributions to M results from (8) to be of the order of a few per cent, D and higher waves thus being negligible. Thus, taking into account only S and P waves, we get:

$$(9)\qquad M = 2c^{(12)}_{00} - \sqrt{3}\{c^{(12)}_{11}(p_{12}, p'_{12})\cos\theta_{12} + c^{(13)}_{11}(p_{13}, p'_{13})\cos\theta_{13}\}\,.$$

The advantage of starting from the decay amplitude in the form (5) is now clear. The resulting expression (9) is only an S-wave in the usual $\pi^+\pi^+$ angular momentum separation, but it shows a linear dependence on p^2_{23} coming from the

(4) B. S. THOMAS and W. G. HOLLADAY: *Phys. Rev.*, **115**, 1329 (1959).

(5) N. N. KHURI and S. B. TREIMAN: preprint. R. F. SAWYER and K. C. WALI: preprint.

$\pi^+\pi^-$ P wave barrier factor. In general, if the coefficient $c_{ll'}(p_{ij}, p'_{ij})$ is of the form $f(p_{ij}^2)p_{ij}^l p_{ij'}^{\prime l'}$ there will be contributions of the $\pi^+\pi^-$ l-waves to all $\pi^+\pi^+$ angular momentum states. Thus, in a self-consistent Dalitz analysis the first correction to the constancy of the $\pi^+\pi^+$ S-wave decay amplitude is the above considered $\pi^+\pi^-$ P-wave contribution.

Introducing instead of the p_{ij}'s, p'_{ij}'s and $\cos\theta_{ij}$'s the energies of the positive and negative pions ε_+ and ε_- (we use non-relativistic kinematics), squaring (9) and multiplying by the phase space factors, we obtain for the differential decay probability

$$\mathrm{d}^2 w = \{1 + \beta(1 - 2\varepsilon_-)\}\,\mathrm{d}\varepsilon_-\,\mathrm{d}\varepsilon_+\,, \tag{10}$$

having conserved in (10) only the S-wave term and the SP interference term. The coefficient β being essentially the P to S-wave ratio,

$$\beta = \frac{2}{9}\cdot\frac{\mu Q}{M^2 c^2}\cdot \mathrm{Re}\,\frac{A_2^{(12)}}{A_0^{(12)}} \tag{11}$$

μ and Q being the pion mass and τ^+-meson Q-value respectively. From (10) follow immediately the energy spectra of the positive and negative pions [6]:

$$\frac{\mathrm{d}w(\varepsilon_+)}{\mathrm{d}\varepsilon_+} = \left(1 + \frac{\beta}{1-\beta/2}\cdot\varepsilon_+\right)\sqrt{\varepsilon_+(1-\varepsilon_+)}\,, \tag{12}$$

$$\frac{\mathrm{d}w(\varepsilon_-)}{\mathrm{d}\varepsilon_-} = \left(1 - \frac{2\beta}{1+\beta}\cdot\varepsilon_-\right)\sqrt{\varepsilon_-(1-\varepsilon_-)}\,, \tag{13}$$

and the angular distribution:

$$\frac{\mathrm{d}w(\theta)}{\mathrm{d}\cos\theta} = 1\,. \tag{14}$$

The agreement of (12), (13) and (14) with the experimental distributions (2), (3) and (4) is excellent, (2) and (3) lead to $\beta = -0.245$ which implies a ratio $m_-/m_+ = -2.97$ for the coefficients of ε_- and ε_+ in (2) and (3), the experimental value being $m_-/m_+ = -(3.1 \pm 0.6)$.

Assuming all the A's in formula (11) to be of the same order of magnitude, this value of β is compatible with a K-meson radius of about 1.4 pion Compton wave lengths, which is a fairly reasonable value.

Thus, the deviations of the experimental energy spectra from those predicted by Dalitz could be a consequence of the presence of $\pi^+\pi^-$ P-waves in the decay amplitude. Strong S-wave $\pi\pi$ final state interaction [1,5] is an alternative explanation.

* * *

I wish to thank Proffs. B. Ferretti, S. Fubini, G. Puppi and A. Stanghellini for many useful discussions.

I am grateful to Dr. J. Tietge for some clarifications on the experimental situation.

[6] S. Weinberg: *Phys. Rev. Lett.*, **4**, 87 (1960). Our formulas (12) and (13) are equivalent to Weinberg's formulas (11) to (17). Weinberg's treatment came to our knowledge as this work had already been completed.

Virtual Pion Effects in Longitudinal Polarization and Spectrum of Neutrons from Unpolarized μ Absorbed at Rest.

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(ricevuto il 1° Giugno 1960)

Several authors [1,4] have considered the problem of strong interactions in the weak processes such as β-decay and μ-capture. Such interactions result in:

a) a renormalization of the axial weak coupling constant;

b) a pseudoscalar term involving the derivatives of the leptonic field;

c) a term of « weak magnetism » [4] from the conserved current hypothesis of Gell-Mann motivated by the lack of renormalization of the vector coupling constant in β-decay.

These virtual pion effects are more appreciable in μ-capture than in β-decay, and the present paper wants to give a first rough estimate of such effects in the spectrum and polarization of the emitted neutrons. Starting from the calculations of Cini and Gatto [5], whose simbols we adopt, a Fermi gas model for the nucleus will be used, considering unpolarized muons absorbed at rest.

An analogous calculation was done by Wolfenstein [3], who however considers only the pseudoscalar term in hydrogen.

Let us assume the proton, the neutron, the neutrino respectively of four-momenta $P_\lambda \equiv (\boldsymbol{P e}, i\mathscr{E})$, $p_\lambda \equiv (p\boldsymbol{n}, iE)$, $h_\lambda \equiv (\boldsymbol{h}, ih)$ and the polarization four-vector of the neutron to be $s_\lambda \equiv (\boldsymbol{\zeta} + (1/M)(E-M)(\boldsymbol{\zeta}\cdot\boldsymbol{n})\boldsymbol{n},\ (ip/M)(\boldsymbol{\zeta}\cdot\boldsymbol{n}))$ where $\boldsymbol{\zeta}$ is the direction of the neutron spin in the neutron's rest system.

The matrix element will be [6]:

$$[\bar{u}_\nu \gamma_\lambda (1+\gamma_5) u_\mu][g_A \bar{u}_n(-\gamma_\lambda\gamma_5 - iCK_\lambda\gamma_5)u_p + g'_V \bar{u}_n(\gamma_\lambda + B\sigma_{\lambda\mu}K_\mu)u_p]\,, \tag{1}$$

(1) J. L. Lopes: *Phys. Rev.*, **109**, 509 (1958).
(2) M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **111**, 354 (1958).
(3) L. Wolfenstein: *Nuovo Cimento*, **8**, 882 (1958).
(4) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); M. Gell-Mann: *Phys. Rev.*, **111**, 362 (1958).
(5) M. Cini and R. Gatto: *Nuovo Cimento*, **11**, 253 (1959).
(6) A. Fujii and H. Primakoff: *Nuovo Cimento*, **12**, 327 (1959).

where

$$g'_V = 0.97 g_V\,, \qquad C = 8/\mu\,, \qquad K_\lambda = P_\lambda - p_\lambda\,, \qquad B = \frac{\mu_p - \mu_n}{2M}\,, \qquad \sigma_{\lambda\mu} = -\frac{i}{2}[\gamma_\lambda, \gamma_\mu]\,,$$

M is the nucleon mass, μ is the muon mass, g_A and g_V are the renormalized coupling constants of β-decay, $\mu_p = 1.793$, $\mu_n = -1.913$ are the proton, neutron, anomalous magnetic moments. Using the Dirac equation one has for the vector part

$$\bar{u}_n(\gamma_\mu + B\sigma_{\lambda\mu}K_\mu)u_p = \bar{u}_n(A\gamma_\lambda + iBQ_\lambda)u_p\,.$$

with $A = \mu_p - \mu_n + 1$ and $Q_\lambda = P_\lambda + p_\lambda$:

From the (1), if the muon is unpolarized and at rest, the distribution takes the form:

$$\frac{W}{2E\mathscr{E}h\mu} = \frac{1}{2E\mathscr{E}h\mu}[W_0 + W_1(\boldsymbol{e}\cdot\boldsymbol{\zeta}) + W_2(\boldsymbol{n}\cdot\boldsymbol{\zeta})]\,,$$

with

$$\begin{aligned}(2)\quad W_0 = \mu g_1^2[(E - \mathscr{E})(\alpha - M^2) + \mu(2E\mathscr{E} + M^2)] + \mu C g_A^2(2M + \alpha C)[\alpha(E - \mathscr{E}) - \mu(E^2 + \mathscr{E}^2) + \\ + \mu(\alpha + 2E\mathscr{E})] + 2\mu g_A g'_V A\alpha(E + \mathscr{E}) + \mu g'^2_V A^2[(E - \mathscr{E})(\alpha + M^2) + \mu(2E\mathscr{E} - M^2)] + \\ + \mu g'^2_V[(2M^2 - \alpha)B^2 - 2ABM][\mu(E + \mathscr{E})^2 + (\mu + \mathscr{E} - E)(\alpha - 2M^2)]\,,\end{aligned}$$

$$\begin{aligned}(3)\quad W_1 = \mu M g_A^2(E + \mathscr{E})P + \mu g_A g'_V[-2AMP\mathscr{E} + 2BP(E + \mathscr{E})(M^2 - \mu E) + \\ + 2ACEP(\mu(E - \mathscr{E}) - \alpha) + 2MBCP(E + \mathscr{E})(\alpha + (\mathscr{E} - E)\mu)] + \\ + \mu g'^2_V[MA^2(\mathscr{E} - E)P - 2ABP(\mathscr{E}M^2 + E(\alpha - M^2))]\,,\end{aligned}$$

$$\begin{aligned}(4)\quad W_2 = \mu g_A^2\beta(E + \mathscr{E}) + \mu g_A g'_V\Big\{2A(2\mu p\mathscr{E} - \mathscr{E}\beta + p\alpha) + \frac{2B}{M}(E + \mathscr{E})[\beta(M^2 - \mu E) + \\ + p\mu(\alpha - 2M^2)] - 2AC[\mu(E - \mathscr{E}) - \alpha][\beta + p(M + \mathscr{E})] + 2BC\beta(E + \mathscr{E})[\alpha + (\mathscr{E} - E)\mu]\Big\} + \\ + \mu g'^2_V\left\{A^2[2p\alpha + (\mathscr{E} - E)\beta] - 2AB\left[\beta\left(\mathscr{E}M + \frac{E}{M}(\alpha - M^2)\right) - \frac{p}{M}((\alpha - M^2)^2 - M^4)\right]\right\}\,,\end{aligned}$$

where $\alpha = M^2 + P_p\xi - E\mathscr{E}$, $\beta = [P\xi(E - M) - p\mathscr{E}]$ with $\xi = (\boldsymbol{e}\cdot\boldsymbol{n})$.

In order to obtain the neutron spectrum and polarization we write the integral

$$(5)\qquad \int \delta(\mu + \mathscr{E} - E - h)\,\delta(\boldsymbol{P} - \boldsymbol{p} - \boldsymbol{h})\,\frac{W}{2E\mathscr{E}\mu h} f(\mathscr{E})[1 - f(E)]\,\mathrm{d}\boldsymbol{P}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{h}\,,$$

where $f(x) = 1$ for $x < F$ and $= 0$ for $x > F$. $F = T_F + M$ with $T_F = 40$ MeV (Fermi energy) (7). From (5) we find that the spectrum is given by

$$U_0\,\mathrm{d}E\,, \qquad \text{with} \qquad U_0 = \frac{1}{2\mu}\int_{\mathscr{E}}^{F} W_0\,\mathrm{d}\mathscr{E}\,,$$

(7) *Handb. d. Phys.*, **39**, 431 (1957).

and the polarization, which is only longitudinal, by

$$P_L = \frac{U_1}{U_0} \qquad \text{with} \qquad U_1 = \frac{1}{2\mu}\int_{\mathscr{E}}^{F} (W_1 \xi_0 + W_2)\,\mathrm{d}\mathscr{E}\,,$$

where ξ_0 is the solution of the equation $\mu + \mathscr{E} - E - \sqrt{P^2 + p^2 - 2pP\xi} = 0$. The integration must be performed from $\mathscr{E}_-$, which is the first solution of the equation of $\mathscr{E}\ \xi_0^2 - 1 = 0$, until F. Using (2), (3), (4), always with Cini and Gatto notations, we have:

$$\begin{aligned}
(6)\quad M^{-4}U_0 = {}& \frac{1}{6}\,a\left[\left(1+\frac{\mu^2C^2}{2}\right)g_A^2 - 2g_A g_V' A + g_V'^2 A^2 + g_V'^2 M^2 B^2\left(4a\varepsilon - 2 - \frac{3}{2}a^2\right)\right](f^3 - u^3) + \\
& + \frac{1}{4}\left\{g_A^2\left(1+\frac{a^2}{2}+\frac{3}{4}a^2\mu^2C^2 - a\mu C - a\mu^2C^2\varepsilon\right) - g_A g_V'^2 A a^2 + g_V'^2\left[A^2\left(\frac{a^2}{2}-1\right) + \right.\right. \\
& \left.\left. + 2(M^2B^2 - ABM)\left(4aE - 2 - \frac{3a^2}{2}\right) + B^2M^2\left((4a+5a^3)\varepsilon - 4a^2\varepsilon^2 - 3a^2 - \frac{5}{4}a^4\right)\right]\right\}(f^2 - u^2) + \\
& + \frac{1}{2}\left\{g_A^2\left[a - \mu C a^2 + \frac{C^2\mu^2a^3}{4} + \left(\mu C a - 1 - \frac{a^2}{2} - \frac{3}{4}C^2\mu^2a^2\right)\varepsilon + \left(a + \frac{aC^2\mu^2}{2}\right)\varepsilon^2\right] + \right. \\
& + 2g_A g_V' A a\varepsilon\left(\varepsilon - \frac{a}{2}\right) + g_V'^2\left[A^2 a(\varepsilon^2 - 1) + A^2\left(1 - \frac{a^2}{2}\right)\varepsilon + 2(M^2B^2 - ABM)\cdot\right. \\
& \left.\left.\cdot\left(\left(\frac{3}{2}a^2 + 2\right)\varepsilon - 2a - \frac{a^3}{2}\right) + M^2B^2\left(\left(\frac{5}{4}a^4 + 3a^2\right)\varepsilon - \left(2a + \frac{3}{2}a^3\right)\varepsilon^2 - a^3 - \frac{a^5}{4}\right)\right]\right\}(f - u)\,,
\end{aligned}$$

$$\begin{aligned}
(7)\quad M^{-4}U_1 = {}& \frac{1}{6}\,\frac{M}{p}\,[(1 - a\varepsilon)(g_A^2 - 2g_A g_V' A - BM\mu C a g_A g_V' + A^2 g_V'^2) + 2g_A g_V' MB(1 + a^2 - 2a\varepsilon) + \\
& + 2AMBg_V'^2(2a\varepsilon - 1 - a^2)](f^3 - u^3) + \frac{M}{4p}\left\{-\frac{a^2}{2}g_A^2\varepsilon + 2Ag_A g_V'\left[\left(1+\frac{a^2}{2}\right)\varepsilon - a\right] + \right. \\
& + 2g_A g_V' MB\left(2a + \frac{a^3}{2} - \frac{a^2}{2}\varepsilon - 2a\varepsilon^2\right) + g_A g_V'\left[\left(A\mu C a + BM\mu C\frac{a^3}{2}\right)\varepsilon - A\mu C a^2\right] + \\
& \left. + g_V'^2\left[A^2\left(2a - \left(2 + \frac{a^2}{2}\right)\varepsilon\right) + 2AMB\left((3a^2+2)\varepsilon - 2a - a^3 - 2a\varepsilon^2\right)\right]\right\}(f^2 - u^2) + \\
& + \frac{M}{2p}\left\{g_A^2\varepsilon^2\left(a\varepsilon - 1 - \frac{a^2}{2}\right) + g_A g_V'\left[2A\,a\varepsilon^3 - \frac{a^2}{2}\varepsilon^2 - a\varepsilon + \frac{a^2}{2}\right) + \right. \\
& + 2MB\left(\left(2a + \frac{a^3}{2}\right)\varepsilon - \left(1 + \frac{3a^2}{2}\right)\varepsilon^2 + AC\mu a\left(a\varepsilon - \varepsilon^2 - \frac{a^2}{2}\right) + BMC\mu a\left(\left(1+\frac{a^2}{2}\right)\varepsilon^2 - a\varepsilon^3\right)\right] + \\
& + g_V'^2\left[A^2\left(a\varepsilon^3 + \left(1 - \frac{a^2}{2}\right)\varepsilon^2 - 2a\varepsilon + a^2\right) + 2ABM\left((2a + a^3)\varepsilon - \right.\right. \\
& \left.\left.\left. - (1 + a^2)\varepsilon^2 - \frac{a^4}{4} - a^2\right)\right]\right\}(f - u)\,,
\end{aligned}$$

with $a=\mu/M$, $f=F/M$, $u=\mathscr{E}_-/M$, $\varepsilon=E/M$. We put $g_A/g_V=-1.21$ (6), and for u it is numerically $u=0.987+0.791(\varepsilon-1)$.

In Fig. 1 we plot the neutron spectrum, and in Fig. 2 $-P_L$ (P_L is negative). $T=E-M-T_F-\Delta$ is the kinetic energy of the neutron outside the nucleus, where Δ is the binding energy (about 8 MeV) of the last neutron. The maximum energy (18 MeV) for the neutron is given by the relation $\mathscr{E}_-(E_{\max})-F$. The dashed curves represent the Cini and Gatto spectrum and polarization, namely with a « pure » $A-V$ coupling, without virtual pion effects. The full curves include these effects following (6) and (7).

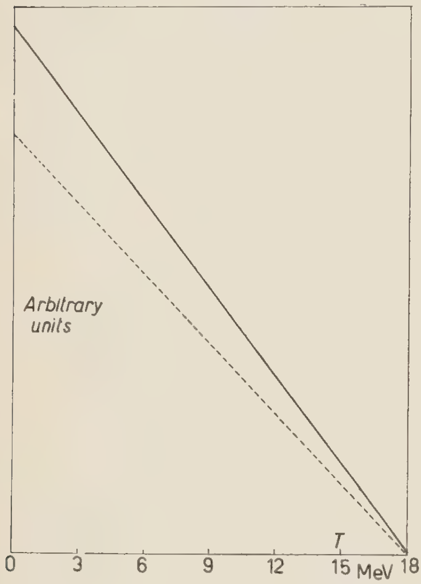

Fig. 1.

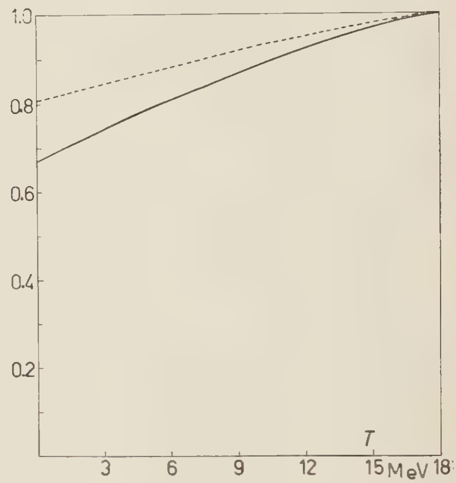

Fig. 2.

As can be seen, the corrections are quite small, particularly at high energies. The evaluated polarization agrees fairly well with the calculations of Wolfenstein, who found smaller corrections using only the pseudoscalar term: it is evident that the « weak magnetism » term gives rise to a slightly larger decrease of the polarization.

As we said before, the results obtained here must be considered in a rather qualitative manner. For more accurate results, it would be necessary to use a more realistic model for the nucleus. The great difference, in « pure » $A-V$, between Dolinskij and Blohinčev (8), who used a shell model in jj coupling, and Cini-Gatto calculations, shows clearly the strong dependence of the process on the nuclear model. The depolarization due to nuclear effects seems to be much larger than the contributions due to virtual pion effects.

To detect such contributions one would need very reliable calculations of the nuclear effects.

* * *

I am very indebted to Prof. M. Cini and Prof. R. Gatto for helpful discussions, and to Drs. N. Cabibbo, E. Ferrari for useful suggestions.

(8) E. I. Dolinskij and L. D. Blohinčev: *Žurn. Èksp. Teor. Fiz.*, **35**, 1488 (1958); L. D. Blohinčev: *Žurn. Èksp. Teor. Fiz.*, **36**, 258 (1959).

Hyperfine Structure of E.S.R. Signals in Phenothiazine Perchlorate (Semiquinone) and Phenothiazine Perbromide (Quinone).

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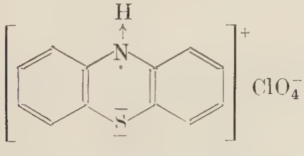
(ricevuto il 3 Giugno 1960)

Recently we have obtained with a E. S. R. spectrometer which has been set up at this Institute, some results on the paramagnetic properties of two phenothiazine compounds, which seem to us of a certain interest.

The two compounds are phenothiazine perchlorate (semiquinone) and phenothiazine perbromide (quinone).

Phenothiazine perchlorate was prepared using the Kehrmann [1] method: it results in a dark-green coloured substance to which the following formula is attributed,



the said form being probably in equilibrium with the dimerous which is diamagnetic. This equilibrium is almost entirely displaced towards the diamagnetic dimerous formula so that only a weak paramagnetism is shown by the substance in magnetostatic measurements [2] and there is only a relatively small E.S.R. signal in the solid state [3].

In further research we have been able to observe the hyperfine structure of the E.S.R. signal.

This structure (Fig. 1*a*) was obtained when phenothiazine perchlorate was dissolved in sulphuric acid ($d = 1.84$).

We think that this structure could be attributed principally to the interaction of the unpaired electron with the nuclear spin of the nitrogen atom; however detailed interpretation asks for more refined work.

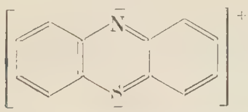
The second substance which we have recently examined, is phenothiazine perbromide also prepared according to the

(1) Kehrmann: *Ber.*, **48**, 325 (1915).

(2) R. Curti: private communication.

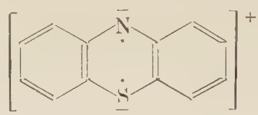
(3) P. Camagni and G. Lanzi: *Rend. 43° Congr. Naz. di Fisica* Padova-Venezia, XII, 4 (1957).

Kehrmann (1) method. This substance of blood-red colour should correspond, following the literature, to the formula:



Br_3^-

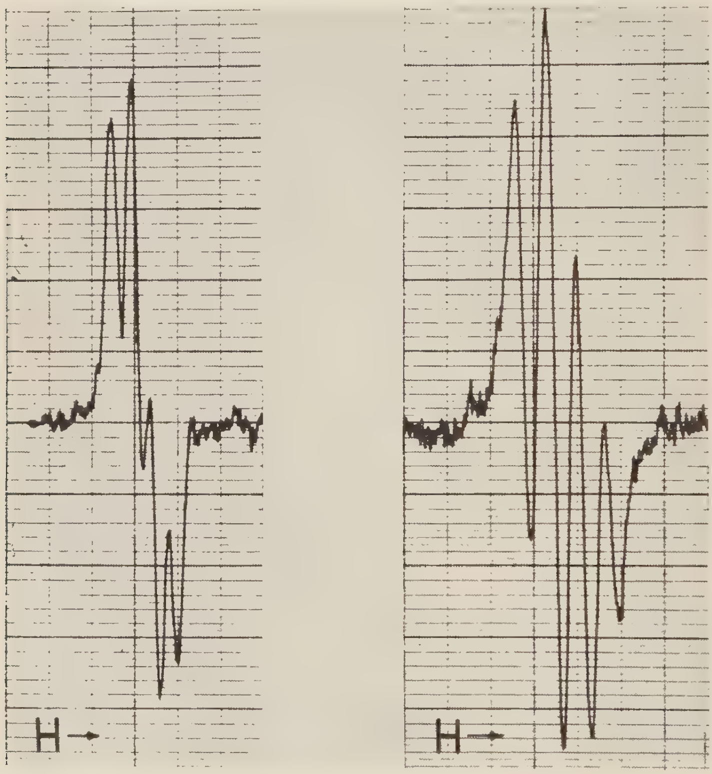
corresponding to the following formula



Br_3^-

or to a possible dimerous form.

The hyperfine structure of this compound has been observed on dissolving the substance in H_2SO_4 (Fig. 1*b*).



a) *b*)

Fig. 1. – Derivatives of E.S.R. signals: *a*) Phenothiazine perchlorate (semiquinone); *b*) Phenothiazine perbromide (quinone).

and therefore should not be paramagnetic.

On the contrary, an analysis of the substance in the solid state by the E.S.R. spectrometer, showed a remarkable unexpected paramagnetism.

The paramegnetism could be justified if one admitted an equilibrium between two forms: one which is the already known form (diamagnetic) and the other

Its structure is less simple than that of phenothiazine perchlorate and its interpretation could provide interesting information regarding the constitution of the actual molecule.

We are now carrying on a systematic investigation in order to check quantitatively the present results.

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We are glad to thank Prof. L. GIULOTTO and Prof. G. CHIAROTTI for many interesting discussions and advice. A special thank is due to Prof. R. CURTI and Dr. S. LOCCHI who proposed the use of these substances and followed their chemical preparation. We also acknowledge the great help given by Dr. E. CROSIGNANI in performing the measurements.

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Some Remarks on the Theory of the Liquid Helium Film.

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(ricevuto il 9 Giugno 1960)

Earlier treatments of the helium film problem, such as that by Schiff [1], taking into account only gravity and the usual r^{-6} van der Waals potential, led to a relation between thickness l and vertical height z above the surface of the bath, having the form

$$al^{-3} = gz \,. \tag{1}$$

Later on, since experiments appeared to support formulae of the type

$$l = \text{const}\, z^{-1/n}\,,$$

with n between 2 and 3, a search was made for terms in l^{-2} in eq. (1). After a suggestion by Bijl, De Boer and Michels [2], which however ran into a very serious objection by Mott [3], two kinds of l^{-2} terms were pointed out almost simultaneously by Atkins [4] and by the writer [5]. Recently Dzyaloshinskij, Lifshitz and Pitaevskij [6] have resumed the problem over again, coming to the conclusion that there can be l^{-3} and l^{-4} terms but no l^{-2} ones.

Here we wish to comment very briefly on the subject. The writer's term is a temperature dependent one vanishing at 0 °K. Although it is probably essentially sound, based as it is on fairly general grounds so as not to be critically dependent on any particular model, let us not discuss it for brevity and simplicity's sake and let us concentrate instead on the larger, temperature independent term pointed out by Atkins.

Atkins derived it by considering the effect of the smallness of l on the zero point energy of the liquid, treating this quantity as the zero-point energy of a system of Debye waves. The same quantity was re-derived in a somewhat more detailed manner by the writer (ref. [5]) and turned out to be

$$E^0 = E^0_\infty \left(1 + \frac{1}{72}\,\lambda_D^2 l^{-2}\right), \tag{2}$$

(1) L. I. Schiff: *Phys. Rev.*, **59**, 839 (1941).
(2) A. Bijl, J. De Boer and A. Michels: *Physica*, **8**, 655 (1941).
(3) N. F. Mott: *Phil. Mag.*, **40**, 61 (1949).
(4) K. R. Atkins: *Can. Journ. Phys.*, **32**, 347 (1954).
(5) S. Franchetti: *Nuovo Cimento*, **5**, 183 (1956).
(6) I. E. Dzyaloshinskij, E. M. Lifshitz, and L. P. Pitaevskij: *Sov. Phys. J.E.T.P.*, **10**, 161 (1960).

where E^0_∞ is the ordinary zero point energy for an extended sample and λ_D is the Debye wave length. The correction term, proportional to l^{-2} is the one which — through the chemical potential — enters eq. (1).

Except for the form, the derivation of ref. (5) was essentially coincident with that by ATKINS (7), the only new feature introduced being a condition of minimum which reduces the correction term somewhat but seems appropriate in dealing with the lowest quantum state of a system (8).

The point we wish to stress about the Atkins term, is that it originates because the system under consideration has a *finite* number f of degrees of freedom. This is particularly evident from the presence in the correction term (eq. (2)) of the factor λ_D^2, which comes from a Debye cut. Of course λ_D would vanish by having f tend to infinity, and so would the l^{-2} term.

Here lies the reason why the l^{-2} term is lacking in the treatment adopted in ref. (6). Indeed the Authors completely disregard the atomistic nature of the liquid and the continuum they consider instead is treated by them in the same way as the « electromagnetic vacuum » in the theory of the attraction between conducting plates by CASIMIR (9). It is true that in this case also a « cutting function » has to be introduced but it has a completely different character from the one needed in the acoustical case. In particular there would be no sense in requiring the cutting function in the electromagnetic case to preserve the « number of degrees of freedom », as is necessary when dealing with a system constituted by a number of atoms. Nor can it be expected with elastic waves that their contribution to the effect should vanish with vanishing wave lenght. Indeed, there is no elastic equivalent for X- or γ-rays.

Questions might of course be raised about the degree of correctness of the Debye cut, but these would seem to be rather academic ones, as the point is that in a way or another one has to reckon with the fact that the system is allowed only a finite number of independent motions. The l^{-2} term arises precisely from that condition, which has no counterpart in the case of a continuum.

In the light of these considerations, the statement in ref. (6), namely, that there are « no physical grounds whatever » to expect l^{-2} terms in the equation of the film profile, appears somewhat unwarranted.

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(7) Indeed, an alternative derivation was also given in ref. (5), but since it does not lend itself to a simple and rigorous discussion, let us ignore it for the moment.

(8) An imperfection might be noted in the writer's derivation and this is to have treated the $3N$ degrees of freedom as *longitudinal* waves [implicit in eq. (6'), ref. (5)]. This of course can be strictly true only for N of these. However, the order of magnitude is not affected and anyway in a simple treatment as the one considered here one has just the choice of leaving part of the zero point energy completely out of account or to treat it imperfectly as longitudinal waves.

(9) H. B. G. CASIMIR: *Proc. Kon. Akad. Wetensch.*, 51, 793 (1948).

LIBRI RICEVUTI E RECENSIONI

M. E. Rose – *Elementary Theory of Angular Momentum*. John Wiley and Sons, Inc., New York, 1957, pp. 248 (della serie *Struttura della Materia*).

Il formalismo del momento angolare — che ha acquistato sempre maggiore importanza nello sviluppo della fisica moderna — è trattato in questo libro in modo semplice ed alquanto esauriente. Senza utilizzare in modo esplicito concetti e linguaggio della teoria dei gruppi, l'A. ha voluto porre l'argomento a portata della maggioranza dei fisici; infatti la preparazione richiesta non va oltre la conoscenza della meccanica quantistica. Questo fatto limita un po' la generalità e persino la semplicità dell'argomento, ma d'altronde risulta necessario se si desidera una trattazione coerente di questi argomenti senza un previo approfondimento di metodi gruppali e di algebra tensoriale.

Il libro è diviso in una parte A, nella quale si sviluppa la teoria, ed una B nella quale essa viene applicata ad alcuni casi particolari. Dopo un breve sunto dei principi basilari, il momento angolare è introdotto come l'operatore generatore di rotazioni infinitesime. Le proprietà di questo operatore sono studiate nel secondo capitolo ottenendone gli autovalori e discutendone l'interpretazione fisica. Nel terzo capitolo si studia il problema dell'accoppiamento di due momenti angolari, vengono così introdotti i coefficienti di Clebsch-Gordan analizzandone le proprietà di simmetria, dandone l'espressione analitica ed ottenendo delle relazioni di ricorrenza per una loro più semplice valutazione. Nel quarto capitolo si studiano le proprietà di trasformazione per rotazione delle autofunzioni. Nel quinto si introducono i tensori irreducibili, studiandone l'algebra, soffermandosi poi con maggior cura sul caso di un campo vettoriale. Nel sesto capitolo si introducono i coefficienti di Racah studiando l'accoppiamento di tre momenti angolari. Nella parte B il formalismo è applicato al campo elettromagnetico, a particelle di spin $\frac{1}{2}$, alle correlazioni angolari in reazioni con nuclei orientati, all'accoppiamento di momenti angolari in reazioni nucleari (distribuzioni angolari) ed allo studio di particelle identiche.

L'elementarietà della trattazione di questi problemi, di per sè tutt'altro che semplici, rendono questo libro di grande utilità, sia ai fisici teorici o sperimentali cui necessiti il formalismo dei momenti angolari nel loro lavoro — ed essi possono senz'altro consultarlo limitatamente — sia, come testo, a coloro che desiderino una conoscenza d'insieme delle tecniche senza voler approfondire molto nella generalità del problema.

D. Amati